KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI

THE FUNDAMENTAL MATRIX OF THE GENERAL RANDOM WALK WITH ABSORBING CHAIN - APPLICATION TO STUDENT FLOW IN AN EDUCATIONAL INSTITUTION

By

FRANCISCA ANANE - ADOMAKO

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Declaration

I hereby declare that this submission is my own work towards the award of the M. Sc degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

Francisca Anane-Adomako	....	....
Student	Signature	Date

Certified by:

Mr. Y. E. Ayekple	....	....
Supervisor	Signature	Date

Certified by:

Prof. S.K. Amponsah	....	....
Head of Department	Signature	Date

Dedication

I dedicate this work to my dear husband Alfred and my dad Mr Patrick Anane-Adomako (Late)

Abstract

This paper presents an application of Markov Analysis of student flow in an educational institution. Historical data of random sample of 1200 students of Kwabenya Atomic Cluster of Schools was investigated. Was first shown that, the Fundamental Matrix of the Markov Analysis can be determined in elementary manner via the adjugate of matrix (I – Q). The Fundamental Matrix of Markov Chain is the inverse of the matrix (I – Q) where Q results after matrix P have been partitioned into canonical form. P is a transition probabilities matrix with entries P_{ij} which satisfy conditions (i) $P_{ij} \geq 0 \quad \text{(ii)} \sum_j P_{ij} = 1$. Results indicated that: (i) a first year student has about 0.93 probability of graduating (ii) first year, second year students stay on average of four (4) terms at their respective levels before they pass on to the next level of study, while senior students stays an average of two (2) terms (iii) A third year student has a probability of one (1) of progressing to graduation stage. Thus, there is a certainty that a student would progress to graduating stage once he/she get to the senior stage. (iv) A percentage of 6.6% of incoming first year students withdraw from their study and (v) the probability of progression to a higher level and graduating increases as students move on to a higher level in the system.

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I also thank all of my friends and colleagues for giving me reasons to persevere and work hard, to laugh, and most of all support through the good times and bad.

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Contents

List of Tables

List of Figures

CHAPTER 1

Introduction

Random walk models have surfaced in various disciplines. They served as initial simple models in biology (especially in genetics) and physics but they are also useful tools in analyzing sequential test procedures in statistics randomized algorithms in computer science (Rudolph, 1999). It can be useful in many real world applications, including: internet applications, economics and finance, insurance, gambling, music, baseball and statistical testing – to name only few fields of application.

Many results have been published for specific instantiations of the transition probabilities; the general case, however, seems to be explored with less intensity. Shafiqah et al (2007) determined the Fundamental Matrix to study the student flow in a higher educational institution but the concept to determine the Fundamental Matrix was not shown. This work aims at a 'closed form' expression for each entry of the fundamental matrix. It will be shown that the fundamental matrix can be achieved via elementary matrix theory. The model will be applied to the Kwabenya Atomic Cluster of Junior High Schools , Ghana, to determine the student flow.

1.1 BACKGROUND OF THE STUDY

The Oxford Advanced Learner's dictionary defines Education as a process of training, teaching and learning, especially in schools or colleges to improve knowledge and develop skills. Education in Ghana requires every student to exhibit some kind of standard performance before one is moved from one stage to another. Over the year's education, has had different goals and after Ghana gained its independence in 1957 the education system (then modeled on the British system) has undergone a series of reforms. Especially the reforms in the 1980.s geared the education system away from purely academic to more in tune with the nations manpower needs (Ministry of Education, 1986).

The present structure of education, on an average, takes about 20 years for a child to complete their education in Ghana. Most children in Ghana begin their education at the age of three or four. They first enter nursery school and then move onto two years in kindergarten. After kindergarten, the child then continues to primary school which starts at the age of 6, which consists of 6 years of primary education, 3 years of Junior Secondary School, 3 years of Senior Secondary School and 4 years University or courses at other tertiary institutions. The first 9 years from the basic education are free and compulsory. In the past, there were more boys enrolled in schools than girls but with the implementation of equal rights for men and women, there are now more number of girls enrolled in schools in Ghana (The Development of Education: National Report of Ghana , 2004). Both boys and girls are expected to meet a criterion before they move to the next grade.

1.1.1 An Overview of the Ga East Municipality

Location and Size

This overview is based on the Ga East Annual District Performance Report for 2012. The Ga East Municipal Directorate of Education was carved out of the then Ga West District in September, 2004 to ensure effective monitoring, supervision and the management of teaching and learning for the overall development of the school-going child.

The Ga East Municipal Directorate is located at the Northern part of Greater Accra Region in a predominantly agricultural area. It is one of the six municipalities in the Greater Accra Region and covers a land area of about 166 square kilometers. It shares boundaries on the west with the Ga West Municipal Assembly (GWMA), on the East with the Tema Municipal Assembly (TMA), on the South with Accra Metropolitan Assembly (AMA), and on the north with the Akwapim South District Assembly (ASDA).

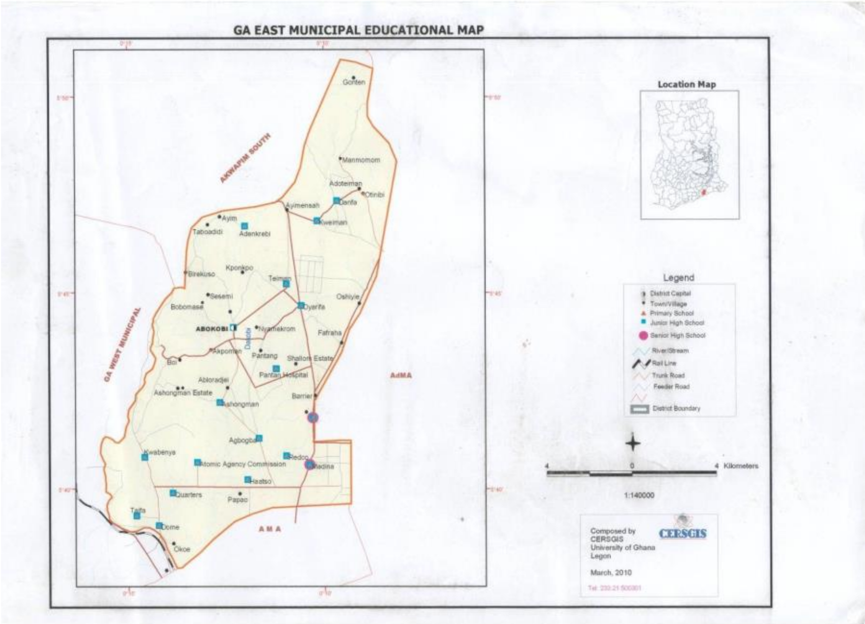

Figure 1.1: Map of Ga East Municipality (source: Ga East Annual District Performance Report, 2012.)

1.1.2 Demographic Characteristics

General Overview of Education in the Municipality

Education infrastructure is distributed quite fairly in the Municipality. The Municipality can boast of two (2) well known Public Senior High Schools namely Presbyterian Boys Senior High School, Legon and West Africa Senior High School.

There are about eleven (11) privately owned Senior High Schools which include: Elim Senior High, Action Progressive institute, Preset Pacesetters Institute, and Faith Baptist Senior High School. There are seventy three (73) Public Junior High Schools and ninety eight (98) Private Junior High Schools. There are also seventy - two (72) Public Primary Schools with about forty (40) Early Childhood Development Centres (ECDC) which enroll only about 10% of children at that age. However, there are three hundred (300) private Early Childhood Development Centres (PECDC).The Directorate has nine (9) circuits namely: Oyarifa, Pantang, Madina, Estate, Presec, Kwabenya, Atomic, Dome and Nkwantanang.

The problems of inadequate and poor quality infrastructure in the public schools can be found throughout the municipality. The situation in the urban areas of the municipality that is Madina, Dome, Haatso, Taifa and Kwabenya is overcrowding with an average of about 100 pupils per class. This means the number of pupils exceeds the number of classrooms and therefore the children are overcrowded with the exception of Adenkrebi which has the lowest enrolment figure. The municipal Education office has put measures in place to improve the delivery of Education.

A number of Tertiary Institutions also exist in the Municipality. These include the Institute of Professional Studies (IPS), Graduate School of Management and Wisconsin University (privately owned). The municipality also shares a common border with the University of Ghana, Legon. There are few private Technical, Vocational Education and Training (TVET) Schools including National Women's Training center of the Department of Community Development at Madina Social Welfare, Ahmadiyah Vocational training center at Agbogba and Lifestyle Vocational Training Institute at Oyarifa.

Ethnic groups

The municipality is predominantly Ga but the inflow of people into the municipality (since the creation of the District in 2004) has made it a mixed ethnic group with a lot of settlers from Northern Ghana, Volta Eastern and Ashanti Regions due to its peri-urban nature. Many people working in government institutions and banks in the capital, Accra reside here.

Economic Activity

Farming and Trading are the predominant economic activities.

Attitude of people towards Education

Generally the people are highly interested in educating their children but are limited by poverty, ignorance and cultural practices especially in a few communities like Otinibi and Adenkrebi.

Political structure

There are two (2) political constituencies namely Abokobi-Madina and Dome–Kwabenya constituencies each with a member of parliament who readily give support to education by using their share of the common fund to assist educational projects. There is tolerance and relative peace in the municipality (Ga East Annual District Performance Report, 2012).

1.2 PROBLEM STATEMENT

The determination of entries of the Fundamental Matrix in the Markov Chain Model becomes more complicated when the size of the matrix increase beyond 4 by 4. There is the need to find simple or an elementary way to determine the Fundamental Matrix and also help other (schools) to use the same model to check the progress of their student performance.

In 2005, the Ministry of Education and Ghana Education Service introduced a system called Computerized Schools Selection and Placement System. This was part of the measures to ensure smooth placement of qualified Basic Education Certificate Exams (BECE) candidates in their various selected secondary schools(Ajayi, 2011). The computerized system uses the total processed raw scores of six subjects representing 4 core and 2 best of each candidate for selection. Indeed, the selection is mainly on the students' performance to be able to get the choice of school. Students who are not able to meet the requirement would not be placed(Mensah et al, 2014). This situation has been a worry to parents, wards and the nation as a whole.

The Junior High Education in Ghana follows a traditional three term calendar. A term consists of an average of 15 weeks and the academic year includes three terms. To complete a Junior High Education in Ghana requires nine (9) terms studies i.e. 3 years, assuming that the student enrolls for every term. The official policy outlining the program of study requires that students attend continuously, realistically; however, many students do not attend for 9 consecutive terms (Basic Curriculum Education: The Junior High Education" Ghana Education Service.) They may lack the intellectual or academic desire necessary to study.

In addition, coming from the primary school, some may face difficulties adjusting to their new environment and regulations at the Junior High Level. Because of these and other social and economic factors, it was decided that a study of student performance at Kwabenya Atomic Cluster of Junior High Schools was needed so that some objective basis would exist to understand and explain the nature of student progression, to evaluate the options available to admission policy and decision makers and to recommend appropriate adjustments to existing policies and procedures pertaining to admissions, transfer, leave of absence, and student advising and counseling services.

1.3 OBJECTIVES OF THE STUDY

The objective of the study has been categorized in to two: main objective and sub-objective. The main objective of the study is to show that the fundamental matrix can be determined in elementary manner via the adjugate of matrix $(I - Q)$.This work is designed to answer the following questions after the fundamental matrix is achieved:

1. At any given time, what is the probability that a student who has been admitted will graduate from the school?

2. At any given time, what is the probability that a student who has been admitted will leave school?

3. What is the average length of time a student will spend in the program at each stage level?

1.3.1 HYPOTHESIS TESTING

Mean or Average Length of Time in School

Statement of Hypothesis

H_0 : The number of terms required by first year students to complete school is nine.

H_1 : The number of terms required by first year students to complete school is not nine.

Mathematically it is given as:

H_0 : $\mu = 9$

H_1 : $\mu \neq 9$

* 9 (From Ministry of Education)

Completion or Graduation Rate

Statement of Hypothesis

H_0 : Graduation rate is same as the national rate.

H_1 : Graduation rate is different from the national rate.

Mathematically it is given as:

H_0 : $p = 0.735$

H_1 : $p \neq 0.735$

* 0.735 (Annual School Census Report, 2015)

Dropout Rate

Statement of Hypothesis

H_0 : Dropout rate is same as the national rate.

H_1 : Dropout rate is different from the national rate.

Mathematically it is given as:

H_0 : $p = 0.048$

H_1 : p 6= 0.048

* 0.048 (Ampiah J.G.,Educational Access in Northern and Southern Ghana)

1.4 METHODOLOGY

The Markov Chain Model will be used for the study. This work seeks to show that the fundamental matrix can be determined in elementary manner via the adjugate of matrix (I – Q) and how it would be applied in Education Sector to determine performance of students in other to help them to get the good grades to continue their education.

School data will be obtained from student's records of Kwabenya Atomic Cluster of Junior High Schools student assessment sheet (cumulative records). Historical data from 2010 to 2012 of students from Kwabenya Atomic Cluster of Junior High Schools will be investigated. Individual student's record containing academic status and grade reports of a random sample of 1200 students will be examined. Thus, students who either completed from or dropped out of the School.

The results of study would be tested against the national figure for dropout rate, completion or graduation rate and the average length of time to complete Junior High School.

1.5 JUSTIFICATION/SIGNIFICANT OF THE STUDY

The significance of this study may be outlined as follows:

1. Help us get less complicated method to determine the entries of the Fundamental Matrix to make the Markov Chain Model more friendly to use.

2. Help get objective basis that would exist to understand and explain the nature of student progression, to evaluate the options available to admission policy

and decision makers and to recommend appropriate adjustments to existing policies and procedures pertaining to admissions, transfer, leave of absence, and student advising and counseling services.

3. The average grade of student would be determined at every level so that student can be monitored in order to have good grade. This would help ease the worries and anxieties that the Computerized Schools Selection and Placement System brings to parents, wards and the nation as a whole.

1.6 ORGANIZATION OF THE STUDY

The study is organized into five chapters, each chapter addressing some main issues on the study. The study begins with the general introduction of the topic 'The Fundamental Matrix of the General Random Walk with Absorbing Chain" and the method used for the study, this constitutes chapter one. This is followed by review of other relevant literature on the Fundamental Matrix of the General Random Walk with Absorbing Chain and a history of Education in Ghana as chapter two of the study. The model used and how the fundamental matrix can be determined in elementary manner via the adjugate of matrix (I – Q) in the study will be discussed in chapter three. Presentation and analysis of data followed in the forth chapter where the data will be thoroughly analyzed. The study ends with the research findings, conclusions and recommendation.

CHAPTER 2

LITERATURE REVIEW

2.1 Education in Ghana

The term education is broad and has variously been described as what happens to us from the day we are born to the day we die. Every Society has its own distinctive ways of educating its members. The fact that, some societies, in the olden days, did not develop the art of literacy (reading and writing) does not mean that no education went on in the society (Oti-Agyen, 2007). Education is therefore defined as the things that are concerned with individuals, their social, physical and spiritual development. Margaret Gillet,(1970) defines education as something concerned "with tradition, preserving and transmitting of values, ideas, practice which have proved over the years to be worthwhile". She further emphasizes that "throughout the age one of the aims of education has been to induct the child into established ways of society. Over the years education has undergone a series of reforms. The Indigenous Ghanaian Education which is highly informal, the Western Formal Education: The Castle Schools System, the British Colonial Government Education and Practice, the Educational Policies under Nationalist Government, the 1887 Educational Reform in Ghana and the Free Compulsory Universal Basic Education (FCUBE) Program (Akyeampong Kwame, 2002). The FCUBE program was launched in 1995 with objectives of improving quality of teaching and learning, improving access to basic education and improving management efficiency(ECOSOC, 2007). Ghana’s long-term vision to become a middle income country by 2020 is clearly stated as "to ensure that all citizens regardless of gender or social status are functionally literate and productive at the maximum" (Ghana Constitution, 1992). The "vision 2020" documented further states that, "the educational system will have primary responsibility for providing the means for the population to acquire the necessary skills to cope successfully in an increasingly global economies"(Ghana Vision 2020). The passage quoted above clearly shows that education should be provided at all cost for all children of school age in order to equip them to become functional literates capable of critical analysis of issues and come out with pragmatic solutions for the problems of under development.

According to Basic Curriculum Education (2014), Education in Ghana consist of Pre-School (Kindergarten), Primary School, Junior High School, Secondary School and Tertiary Education. Most children in Ghana begin their education at the age of three or four. They first enter nursery school and then move onto two years in kindergarten. After kindergarten the child then continues to primary school which starts at the age of 6 years, which consists of 6years of primary education, 3 years of Junior Secondary School, 3 years of Senior Secondary School and 4 years University or courses at other tertiary institutions. The first 9 years from the basic education and are free and compulsory. In the past, there were more boys enrolled in schools than girls but with the implementation of equal rights for men and women, there are now more number of girls is enrolled in schools in Ghana. Both boys and girls are expected to meet a criterion before they move to the next grade (Olivia Asselin , 2007).

2.2 Basic Education

Formal education was introduced to modern Ghana by the European merchants. In 1529, the first of these merchants (Portuguese) established the first school in Ghana (Graham, 1974). From that time, education has undergone reforms with different aims and goals. Basic education in Ghana consist of three levels; the Pre-School (Kindergarten), Primary and Junior High level. The present structure of education, takes 9 years for a child to complete their basic education in Ghana. Most children in Ghana begin their education at the age of three or four. They first enter Pre-School at supposed age of four (4) for two (2) years. After kindergarten the child then continues to primary school which starts at the age of 6 years, which consists of 6years of primary education, 3 years of Junior Secondary School.

To be able to achieve the vision 2020, Ghana government introduced policies to help achieve the vision, from 1995 to date, the Government of Ghana has been building many classrooms in various communities, even at very remote rural areas in some part of Ghana to improve access of basic education. Free text books and teachers guide and other teaching aids are being provided to pupils and teachers to improve quality of teaching and learning in Ghana. In 2006, Free School Feeding Program was introduced by the Government of Ghana to improve enrolment drive (Martens, 2007). School uniforms are provided to pupils in order to help achieve vision 2020.

The curriculum of elementary schools was made up of English, Reading and Writing, Akuapim Twi and Ga, Arithmetic as well as Bible Studies. Currently it comprises the subject in the Basic Education Certificate Examination which qualifies one to the Secondary School. This examination has to be taken before a student is accepted into Senior High School. Formerly, it used to cover 10 subjects ranging from: Mathematics, English, Social studies, General science, Agricultural science, Pre-technical education, Pre-vocational education, Religious and Moral Education and French. But as of 2010, most of these subjects have been integrated and changed whilst others have been dropped and new ones added. The examinable subjects are now: Mathematics, English, Social Studies, Integrated Science, Basic Designing and Technology (candidates can choose between Pre-technical Skills, Home Economics and Visual Arts), Information Communication and Technology, Religious and Moral Education and French.

2.3 Secondary Education

Students who pass the BECE are accepted into Senior High School. Senior Secondary School is the same as High School in America, it takes a student three years to complete Senior Secondary School but the system had been reformed in 2008, to be a four year term for completion (A Brief History of the Ghanaian Educational System, 2011). However, early 2009 this reform was immediately reversed again by the new National Democratic Congress government government, and presently it is 3 years again. Elective courses offered are:

- General Arts I (consists of subjects ranging from economics, calculus I and II, geography and French).
- General Arts II (consists of subjects ranging from literature, trigonometry and pre-calculus, history and French).
- Agriculture (consists of subjects ranging from chemistry, physics, agricultural science and calculus I and II)
- Business (consists of subjects ranging from accounting, business management, calculus I and II)
- Science (consists of subjects ranging from biology, chemistry, physics and calculus I and II)

In the third year students write their final exam called the West African Senior School Certificate Examination (WASSCE). It consists of subjects from the elective courses.

2.4 Markov Chain

A state X_i of a Markov chain is called an absorbing state if, once the Markov chains enter the state, it is impossible to leave that state (Konstantopoulos, 2009). Therefore the probability of leaving that state would be zero and it is shown as $p_{ii} = 1$. A Markov chain is absorbing if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state. In an absorbing Markov chain, a state which is not absorbing is called transient. In the example below, 0 and 4 are in the absorbing state with the probability of 1. Therefore the chain is an absorbing chain. States 1, 2 and 3 are transient.

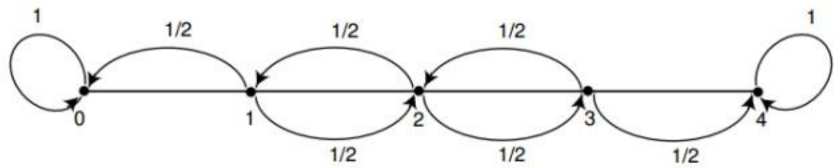

Figure 2.1: Absorbing and Transient States. (After Carroll et al, 2012)

In an absorbing Markov chain if the beginning state is absorbing, we will stay at that state and if we start with a transient state, we will eventually end up at an absorbing state.

2.4.1 Applications of Markov Chains

Markov chains can be useful in many real world applications, including: internet applications, economics and finance, gambling, music, baseball and statistical testing (Holt, 2010). He focused on how Markov chains can be used to find the probability of being absorbed into an ergodic state given that it starts out in a transient state. He applied the properties of Markov chains to a specific example about a cat and mouse in a box separated into 3 rooms to find out the probability that they will end up in the same room. He also used simulations to verify these probabilities.

El–Shehawey and Trabya (2006) have determined the joint probability generating function of the number of occurrences of the transient states. Its marginal's may be used to derive the fundamental matrix but the expression offered contains unresolved recurrence relations that make potential further calculations difficult.

A closed form of expressions for the entries of the fundamental matrix (Fundamental Matrix in Markov Chain) of the general random walk with absorbing boundaries have been derived by means of elementary matrix theory. (Rudolph, 1999). However, the approach taken here is especially useful if for example, the absorption time or the absorption probabilities for a specific initial state are of interest because only few entries of the fundamental matrix will be determined.

Shafiqah et al (2007) and Bessent & Bessent (1980) determined the Fundamental Matrix to study the student flow in a higher educational institution but the concept to determine the Fundamental Matrix was not shown.

The theory behind Markov chains and how it can be used to solve certain applications. Using Markov chains we can solve the question," Does this system lead to absorption...and how long does it take.” (Holt, 2010). He focuses his work on how Markov chains can be used to find the probability of being absorbed into an ergodic state given that it starts out in a transient state. He applied the properties of Markov chains to a specific example about a cat and mouse in a box separated into 3 rooms to find out the probability that they will end up in the same room.

The application of Absorbing Markov’s Chain, the use of Decision Support Systems and Non Linear Programming in relations to the board game Chutes and Ladders. The game of Chutes and Ladders can be studied as a Markov Chain with one absorbing state, the winning state (Gonzalez, 2011). He use of transition matrices and Markov’s Chain Fundamental Matrix to calculate the expected number of turns that a player needs to go through to win the game. In order to easily generate transition matrices and Markov’s Chain Fundamental Matrix a Decision Support System (DSS) was created using Microsoft’s Excel
Visual Basic. The DSS lets a user input the specific attribute of a Chutes and Ladders Board game and calculate the changes in expected number of turns to win the game. Non Linear Programming is used to determine the best strategy to win the game if the players have two spins per turn, instead of one. The applications and processes

utilized in this study can be applied to areas like manufacturing, supply chain, and economic analyses

Coppens and Verduyn (2009), used the theory of finite Markov chains to analyse the demographic evolution of Belgian enterprises. They used Markov Chain to compute the average lifetimes of companies by branch of activity. Kwak, (1986) on his part gave an explanation, application and discussion of the use of Markov analysis in forecasting student enrollment variations for a "real-world" academic institution. The results also illustrated the usefulness of Markov analysis in forecasting student enrolments and suggested extensions for further research.

Haan, el at (1976), developed a simulated daily rainfall at a point using Markov chain. The model was able to simulate a daily rainfall record of any length, based on the estimated transitional probabilities and frequency distributions of rainfall amounts. It was realized that the simulated data have statistical properties similar to those of the historical data. Adeleke el at (2014), studied the pattern of students' enrolment and their academic performance in the Department of Mathematical Sciences (Mathematics Option) Ekiti State University. The probabilities of absorption (Graduating and Withdrawal) were the findings. They also generated and used fundamental matrix to determine the expected length of students' stay before graduating.

CHAPTER 3

METHODOLOGY

This section presents the conceptual framework of the study and to show that the fundamental matrix can be determined in element manner via the adjugate of matrix (I – Q).

3.1 General Random walk

Suppose that a particle is initially at a given point X_0 on the x-axis. At time $k = 1$ the particle undergoes a step or jump Z_1 where Z_1 is a random variable having a given distribution. At time $k = 2$, the particle undergoes a jump Z_2 where Z_2 is independent of Z_1 and with same distribution, and so on. Thus the particle moves along a straight line and after one jump is at $X_0 + Z_1$, after two jumps at $X_0 + Z_1 + Z_2$ and in general, after k jumps, i.e. at time k, the position of the particle is given by

$$X_k = X_0 + Z_1 + Z_2 + \dots + Z_k$$

Where Z_i is a sequence of mutually independent, identical distribution random variables. We say that the particle undergoes a general one dimensional random walk or brief a random walk. Alternatively we may write as

$$X_k = X_{(k-1)} + Z_n (n = 1,2,\dots)$$

In the particular case where the steps Z_i can only take the value 1, 0 or -1 with the distribution

$$prob(Z_i = 1) = p, \qquad prob(Z_i = 0) = 1 - p - q, \qquad prob(Z_i = -1) = q$$

We shall call the process a simple random walk. It is usual in the literature to define a simple random walk as one for which each jump is either +1 or -1 with $p + q = 1$ However, we shall assume that $p + q \leq 1.$ with $1 - p - q$ as the probability of a zero jump. The extra generality is achieved with hardly any cost in analytical computation (Bhat, 1980).

According to the definition, the random walk is a stochastic process in discrete time. Thus, the state space is considered continuous if the steps Z_i are continuous random variables and discrete if the steps are restricted to integral values in the simple random walk. If a particle continuous to move indefinitely, we say the random walk is unrestricted. However, we consider the motion of the particle to be restricted by the presence of barriers. For example, a random walk starting at $X_0 = 0$ may be restricted to within a distance up and b down from origin in such a way that when the particle reaches or overreaches either of the points a or –b, the motion ceases. The points a and –b are absorbing barriers in this case and the states $x \geq a$ and $x \leq -b$ are all absorbing state.

In this study we would look at special case of random walk on the nonnegative integer with absorbing boundaries. This special case is known as the Absorbing Markov Chain.

3.2 Markov Chain

A Markov chain is a finite process that models a sequence of events that has a fixed number of states and their specified probabilities, and given the present state the future and past states is independent (Holt, 2010). At each step the system may change its state from the current state to another state or it may stay in the same state according to a probability distribution. These changes of state are called transitions, and the probabilities that are associated with various state changes are called transition probabilities. In simpler terms a Markov chain is just a random process or series of events that occur by chance and evolve in time with different probabilities. These random processes or series do not depend on future or past states.

Tim Carroll et al (2012), also defined Markov chain as a discrete-time process. This means assuming we have a set of states:

$$S = S_1, S_2, \ldots, S_r$$

The process can start in one of these states and move to another state. Each move is called a step. Each step has a probability of its own. If the chain is currently in state S_i, then it moves to state S_j at the next step with a probability shown by P_{ij}, and this probability does not depend upon which states the chain was in before the current. The probabilities pij are called transition probabilities. The probabilities can be shown in a matrix called transition matrix. In the P transition matrix below P_{ij} is the probability of being in state S_i at step $n + 1$ given that the process was in state S_j at step r.

$$P = \begin{pmatrix} p_{11} & p_{12} & \ldots & p_{1r} \\ p_{21} & p_{22} & \ldots & p_{2r} \\ \ldots & & & \ldots \\ p_{r1} & p_{r2} & \ldots & p_{rr} \end{pmatrix}$$

Here is an example of a process with r states:

$$\text{State } n + 1 = \begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array} \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

Also, Q_i is the probability of the chain at time 0. The entries in the matrix P are nonnegative and sum of each row equals to 1. For a Markov chain with a transition matrix P, $p^n{}_{ij}$ gives the probability that the Markov chain, starting in state s_i, will be in state s_j after n step.

A Markov chain can again be defined as a mathematical model of a random phenomenon evolving with time in a way that the past affects the future only through the present (Konstantopoulos, 2009). It is a type of discrete – time stochastic process and assume that, at any time, the discrete – time stochastic process can be in one of the finite of states 1,2,...,n.

Definition: A discrete – time stochastic process is called a Markov Chain, if for k = 0,1,2... and all states

$$P = (X_{(k+1)} = i_{(k+1)} | X_k = i_k, \quad X_{(k-1)} = i_{(k-1)}, \ldots, X_0 = i_0) = P(X_{(k+1)} = i_{(k+1)} | X_k = i_k)$$

Essentially, equation above means that, the probability distribution of the state at time $k + 1$ depends on the state at time $k(i_k)$ and does not depend on the states the chain passed through on the way to i_k at time k. we make assumption that, for all state i and j and all k.

$P(X_{k+1} = j | X_k = i)$ is independent of time. This assumption allow us to write

$$P(X_{k+1} = j | X_k = i) = P_{ij}$$

where P_{ij} is the probability that given the system j at time $k + 1$. The P_{ij} are the transition probabilities for the Markov Chain.

We consider a type of discrete – time stochastic process with set $Z = \{0,1,2 \ldots\}$; that is, we have a sequence $\{X_k : k \in Z\}$ of random variables. The subscript k in X_k stands for the time and X_k denotes the state of the process at time k. These stochastic processes that we consider here satisfy the Markov property. Definition: A discrete – time stochastic process is called a Markov Chain, if for $k = 0,1,2...$ and all states

$$P = (X_{k+1} = i_{k+1}|X_k = i_k, X_{k-1} = i_{k-1}, ..., X_0 = i_0) = P(X_{k+1} = i_{k+1}|X_k = i_k) \tag{3.1}$$

Essentially, equation 3.1 means that, the probability distribution of the state at time $k + 1$ depends on the state at time $k(i_k)$ and does not depend on the states the chain passed through on the way to i_k at time k. We make assumption that, for all state i and j and for all k.

$P(X_{k+1} = j|X_k = i)$ is independent of time. This assumption allow us to write

$$P(X_{k+1} = j|X_k = i) = P(X_1 = j|X_0 = i) = P_{ij}$$

where P_{ij} is the probability that given the system j at time k+1, that means P_{ij} determines the probability of moving from state i to j in just one transition and all the quantities define the matrix of one-step transition probability P.

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1r} \\ p_{21} & p_{22} & \cdots & p_{2r} \\ \vdots & & \ddots & \vdots \\ p_{r1} & p_{r2} & \cdots & p_{rr} \end{pmatrix}$$

where the finite set $I_r = 1,2,...,r$ is called the state space of the Markov Chain.

The entries p_{ij} of the matrix must satisfy:

$$(i) \quad P_{ij} \geq 0 \qquad (ii) \sum_j P_{ij} = 1 \qquad i,j \in I_r$$

3.2.1 Transition and Recurrent State

A state i is said to be transient if it can be reached only finitely many times or if there is a way to leave state i that never returns to state i. A state is said to be recurrent if it can be reached infinitely often. If a state is not transient, it is called recurrent state.

3.2.2 Absorbing State

A state i is an absorbing state once it enters the state, it is impossible to leave that state. A recurrent state is called absorbing if there is no way to escape it once it is entered. Therefore the probability of leaving that state would be zero and it is shown as $p_{ii} = 1$.

3.3 Canonical Form

Based on the classifications of the state of being transient or recurrent (absorbing and non-absorbing) the transition matrix P can be partitioned into its canonical form, which means that the transient states come first:

$$\mathbf{P} = \begin{array}{c} \\ \text{TR.} \\ \text{ABS.} \end{array} \begin{array}{c} \begin{array}{cc} \text{TR.} & \text{ABS.} \end{array} \\ \left(\begin{array}{c|c} \mathbf{Q} & \mathbf{R} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right) \end{array}$$

Figure 3.1: Canonical form of the Transition Probability matrix P (After Carroll et al, 2012)

If we have t transient states and r absorbent states, then:

If we have t transient states and r absorbent states, then: I: is an r-by-r identity matrix, reflecting the fact that we never leave an absorbing state. 0: is a r -by-t zero matrix, reflecting the fact that it is impossible to move from absorbing state to transition state.

R: is a nonzero t-by-r matrix, giving transition probabilities from transient state to absorbing states

Q: is a t-by-t matrix, giving transition probabilities from transition to transition states.

3.3.1 The Fundamental Matrix

Theorem: For an absorbing Markov chain the matrix $I - Q$ has an inverse B and

$B = I + Q + Q^2 + ...$

For an absorbing Markov chain P, the matrix $B = (I - Q)^{-1}$ is called the fundamental matrix for P. The entry b_{ij} of B gives the expected number of times that the process is in the transient state X_j if it is started in the transient

state X_i. Define an indicator random variable $I_{(X_n=j)}$ which takes value 1 if $X_n = j$ and 0 if X_n 6= j. It is known that for two transient states i and j, the ij^{th} entry of B_{ij} the matrix B is

$$B_{ij} = E_i \sum_{n=0}^{\infty} I_{(X_n=j)}$$

where the right hand side denotes the mean number of visits to the state j having started at state i.

Define $\tau = inf\{n \geq 0 : X_n \in T\}$ to be exit time of T (the time to leave T). For an absorbing Markov chain, τ is the time of absorption. Let

$M_i = E_i\tau$ denote the mean time of absorption starting at state i. Let M stands for the vector whose components are $M_i = i = 1,2,\ldots,k$. It is known that

$$M = B\varepsilon$$

where ε is the vector that all of its components are 1.

For an absorbing Markov Chain, t is the time of absorption. Let $T_i = E_i t$ denote the mean time of absorption starting at i. Let T stands for the vector whose components are T_i, $i = 1,2,\ldots,$k.

3.3.2 Absorption Probabilities

If B is a t-by-r matrix with entries b_{ij} and b_{ij} is the probability that an absorbing chain will be absorbed in the absorbing state X_j if it starts in the transient state X_i. Then:

$$N = BR$$

Where B is the fundamental matrix and R is as in the canonical form and N is the Absorbing Probabilities.

Properties of Transient and Absorbing states

As stated before, Absorbing states cannot revert to any other state at any time. This means that the chance of the state remaining the same is equal to 1, or 100%. While this is the only true property of Absorbing states, Transient states can take many shapes:

1. Transient States may only Progress forward until reaching an Absorbing State
2. Transient States may skip other transient states and go directly into an Absorbing State
3. Transient states may revert back to previous transient states before reaching Absorbing
4. Transient States have the chance to stay the same, however, the chance of this being true is not equal to 1, hence the following:
   a. If ta goes to ta in element $< a,a >$ and P 6= 1 then Transient
   b. If ta goes to ta in element $< a,a >$ and $P = 1$ then Absorbing

3.4 Markov Chain Model

The general random walk with absorbing boundaries is a time homogeneous Markov chain $(X_k : k \geq 0)$with state space 0,1,...,n and the transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & \dots & & & 0 \\ p_1 & r_1 & q_1 & \dots & & & 0 \\ 0 & p_2 & r_2 & q_2 & \dots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & p_{n-2} & r_{n-2} & q_{n-2} & 0 \\ 0 & \dots & & 0 & p_{n-1} & r_{n-1} & q_{n-1} \\ 0 & \dots & & & 0 & 0 & 1 \end{pmatrix}$$

such that $P\{X_{k+1} = j|X_i = i\} = P_{ij}$ for $i,j = 0,1,...,n$. Let $Q = P(0,n)$, i.e. Q results from P by deleting the rows and columns 0 and n, and set $A = I - Q$. Then $B = A^{-1}$ is the fundamental matrix associated with the transition matrix P. Let $T = min\{k \geq 0 : X_k \in \{0,n\}\}$. Then $E[T|X_0 = i] = a_i$ denotes the absorption time for a random walk starting at state i where a_i is the ith entry of vector $a = B_e$. Thus a_i is just the sum of all entries of row i of the fundamental matrix. In case of the random walk, the absorption probabilities are

$P\{X_T = 0|X_0 = i\} = b_i.p_10$ and $P\{X_T = n|X_0 = i\} = b(i,n-1).p(n-1,n)$ for $i = 1,...,n = 1$.

Notation

Let A be an mxn matrix. Then $A(\alpha_1,...,\alpha_h|\beta_1,...,\beta_k)$ denotes the $(m-h)x(n-k)$ sub matrix of A obtained from A by deleting rows $\alpha_1,...,\alpha_h$ of A and column $\beta_1,...,\beta_k$ where as $A[\alpha_1,...,\alpha_h|\beta_1,...,\beta_k]$ denotes the hxk sub matrix of A whose $A(i,j)$ entry is $\alpha_{\alpha_i,\beta_j}$. If $\alpha_i = \beta_j$ for $i = 1,...,k$ then the shorthand notation $A(\alpha_1,...,\alpha_k)$ resp. $A(\alpha_1,...,\alpha_k)$ will be used. As usual, A^{-1} is the inverse and detA is the determinant of a regular square matrix A. Matrix I is the unit matrix and every entry of column vector e is 1.

3.4.1 Determination of the Fundamental Matrix

There are many methods to obtain the inverse of some regular square matrix. Here, the inverse of matrix (I – Q) is determined via its adjugate. This approach is especially useful if only few elements of the inverse are of interest.

Let A: $d \times d$ be a regular square matrix. The adugate adj(A) of matrix A is the matrix whose (i,j) entry is $(-1)^{j+i}\det A(j|i)$. Since $B = A^{-1} = adj(A/det(A))$ one obtain

$$b_{ij} = (-1)^{i+j}\frac{\det A(j|i)}{\det A}$$

for $i,j = 1,\ldots,d$. To proceed, one needs an elementary expression for the determinant for matrix A.

Let P be the transition matrix of the general random walk with absorbing boundaries at state 0 and state n. Let $Q = P(0,n)$ and set $A_d = I - Q$ with $d = n - 1$. The determinant of A_d is give by.

$$detA_d = \sum_{k=0}^{d}\left(\prod_{i=1}^{d-k} p_i\right)\left(\prod_{j=d-k+1}^{d} q_j\right.!$$

for all $d > 1$.

Proof (by induction)

Let d = 1,

For the left hand side,

$$A_d = (I - Q)$$

but $\quad p_1 + r_1 + q_1 = 1$

$$A_1 = (1 - r_1) = p_1 + q_1$$

$$detA_1 = det(p_1 + q_1) = p_1 + q_1$$

For the right hand side,

$$\sum_{k=0}^{d}\left(\prod_{i=1}^{d-k}p_i\right)\left(\prod_{j=d-k+1}^{d}q_j\right)=\sum_{k=0}^{1}\left(\prod_{i=1}^{1-k}p_i\right)\left(\prod_{j=2-k}^{1}q_j\right)$$

$$=\left(\prod_{i=1}^{1}p_i\right)\left(\prod_{j=2}^{1}q_j\right)+\left(\prod_{i=1}^{0}p_i\right)\left(\prod_{j=1}^{1}q_j\right)$$

by convention

$$\prod_{i=k}^{r}p_i=1 \qquad \text{for } k>r$$

$$p_1(1)+(1)q_1$$

$$\sum_{k=0}^{1}\left(\prod_{i=1}^{1-k}p_i\right)\left(\prod_{j=2-k}^{1}q_j\right)=p_1+q_1$$

The hypothesis for $d=1$ is true.

Let $d=2$,

For the left hand side,

$$A_2=\begin{pmatrix}1-r_1 & -q_1\\ -p_2 & 1-r_2\end{pmatrix}$$

$$\det A_2=\det\begin{pmatrix}1-r_1 & -q_1\\ -p_2 & 1-r_2\end{pmatrix}=\det\begin{pmatrix}p_1+q_1 & -q_1\\ -p_2 & p_2+q_2\end{pmatrix}$$

$$\det A_2=(p_1+q_1)(p_2+q_2)-p_2q_1$$

$$\det A_2=p_1p_2+p_1q_2+p_2q_1$$

for the right hand side,

$$\sum_{k=0}^{2}\left(\prod_{i=1}^{d-k}p_i\right)\left(\prod_{j=d-k+1}^{d}q_j\right)=\sum_{k=0}^{2}\left(\prod_{i=1}^{2-k}p_i\right)\left(\prod_{j=3-k}^{2}q_j\right)$$

$$= \left(\prod_{i=1}^{2} p_i\right)\left(\prod_{j=3}^{2} q_j\right) + \left(\prod_{i=1}^{1} p_i\right)\left(\prod_{j=2}^{2} q_j\right) + \left(\prod_{i=1}^{0} p_i\right)\left(\prod_{j=1}^{2} q_j\right)$$

by convention $\prod_{i=k}^{r} p_i = 1$ for $k > r$

$$\sum_{k=0}^{2}\left(\prod_{i=1}^{2-k} p_i\right)\left(\prod_{j=3-k}^{2} q_j\right) = p_1p_2 + p_1q_2 + p_2q_1$$

The hypothesis is true for $d = 2$ as well. Suppose that the hypothesis is true for $d - 1$ and $d \geq 2$. Thus,

$$detA_d = \sum_{k=0}^{d}\left(\prod_{i=1}^{d-k} p_i\right)\left(\prod_{j=d-k+1}^{d} q_j\right)$$

$$detA_{d-1} = \sum_{k=0}^{d-1}\left(\prod_{i=1}^{d-1-k} p_i\right)\left(\prod_{j=d-k}^{d-1} q_j\right)$$

we show that

$$detA_{d+1} = \sum_{k=0}^{d+1}\left(\prod_{i=1}^{d+1-k} p_i\right)\left(\prod_{j=(d+1)-k+1}^{d+1} q_j\right)$$

we know that

$$A_1 = p_1 + q_1$$

$$A_2 = \begin{pmatrix} p_1 + q_1 & -q_1 \\ -p_2 & p_2 + q_2 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} p_1 + q_1 & -q_1 & 0 \\ -p_2 & p_2 + q_2 & -q_2 \\ 0 & -p_3 & p_3 + q_3 \end{pmatrix}$$

...

$$\det A_{d+1} = \det\begin{pmatrix} \boxed{A_d} & \vdots \\ & -q_d \\ -p_{d+1} & p_{d+1}+q_{d+1} \end{pmatrix}$$

A_{d+1} can be again be written as

$$A_{d+1} = \begin{pmatrix} \boxed{A_{d-1}} & & -q_{d-1} \\ -p_d & p_d+q_d & -q_d \\ - & p_{d+1} & p_{d+1}+q_{d+1} \end{pmatrix}$$

The determinant of matrix expressed in terms of det A_d and det A_{d+1} as

$$\det A_{d+1} = \det\begin{pmatrix} \boxed{A_d} & \vdots \\ & -q_d \\ -p_{d+1} & p_{d+1}+q_{d+1} \end{pmatrix}$$

$$\det A_{d+1} = \det\begin{pmatrix} \boxed{A_{d-1}} & & -q_{d-1} \\ -p_d & p_d+q_d & -q_d \\ & -p_{d+1} & p_{d+1}+q_{d+1} \end{pmatrix}$$

$$= (p_{d+1}+q_{d+1})\det A_d + p_{d+1}\det\begin{pmatrix} \boxed{A_d} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ -p_d & -q_d \end{pmatrix}$$

$$= (p_{d+1}+q_{d+1})\det A_d - p_{d+1}q_d\det(A_{d-1})$$

$$= p_{d+1}\det A_d + q_{d+1}\det A_d - p_{d+1}q_d\det A_{d-1}$$

$$= p_{d+1}(detA_d - q_d detA_{d-1}) + q_{d+1}detA_d$$

For $q_{d+1}\, det\, A_d$,

$$q_{d+1}detA_d = q_{d+1}\sum_{k=0}^{d}\left(\prod_{i=1}^{d-k} p_i\right)\left(\prod_{j=d-k+1}^{d} q_j\right)$$

$$= \sum_{k=0}^{d}\left(\prod_{i=1}^{d-k} p_i\right)\left(q_{d+1}\prod_{j=d-k+1}^{d} q_j\right)$$

$$= \sum_{k=0}^{d}\left(\prod_{i=1}^{d-k} p_i\right)\left(\prod_{j=d-k+1}^{d+1} q_j\right)$$

$$= \sum_{k=1}^{d+1}\left(\prod_{i=1}^{d+1-k} p_i\right)\left(\prod_{j=(d+1)-k+1}^{d+1} q_j\right)$$

For $q_d\, det\, A_{d-1}$

$$q_d detA_{d-1} = q_d\sum_{k=0}^{d-1}\left(\prod_{i=1}^{d-1-k} p_i\right)\left(\prod_{j=d-k}^{d-1} q_j\right)$$

$$= \sum_{k=0}^{d-1}\left(\prod_{i=1}^{d-1-k} p_i\right)\left(q_d\prod_{j=d-k}^{d-1} q_j\right)$$

$$= \sum_{k=0}^{d-1}\left(\prod_{i=1}^{d-1-k} p_i\right)\left(\prod_{j=d-k+1}^{d} q_j\right)$$

$$= \sum_{k=0}^{d}\left(\prod_{i=1}^{d-k} p_i\right)\left(\prod_{j=d-k+1}^{d} q_j\right)$$

$p_{d+1}(det\, A_d - q_d\, det\, A_{d-1})$

$$= p_{d+1}\left[\sum_{k=0}^{d}\left(\prod_{i=1}^{d-k} p_i\right)\left(\prod_{j=d-k+1}^{d} q_j\right) - \sum_{k=1}^{d}\left(\prod_{i=1}^{d-k} p_i\right)\left(\prod_{j=d-k+1}^{d} q_j\right)\right]$$

$$= p_{d+1}\left[\left(\prod_{i=1}^{d} p_i\right)\left(\prod_{j=d+1}^{d} q_j\right) + \sum_{k=1}^{d}\left(\prod_{i=1}^{d-k} p_i\right)\left(\prod_{j=d-k+1}^{d} q_j\right) - \sum_{k=1}^{d}\left(\prod_{i=1}^{d-k} p_i\right)\left(\prod_{j=d-k+1}^{d} q_j\right)\right]$$

$$= P_{d+1}\left(\prod_{i=1}^{d} p_i\right), \quad \text{since} \quad \left(\prod_{j=d+1}^{d} q_j\right) = 1$$

$$p_{d+1}(det\, A_d - q_d\, det\, A_{d-1}) = \prod_{i=1}^{d+1} p_i$$

$$p_{d+1}(det\, A_d - q_d\, det\, A_{d-1}) + q_{d+1}\, det\, A_d = \prod_{i=1}^{d+1} p_i + \sum_{k=1}^{d+1}\left(\prod_{i=1}^{d+1-k} p_i\right)\left(\prod_{j=(d+1)-k+1}^{d+1} q_j\right)$$

$$\sum_{k=0}^{d+1}\left(\prod_{i=1}^{d+1-k} p_i\right)\left(\prod_{j=(d+1)-k+1}^{d+1} q_j\right)$$

Which is the desired result. We need to find the elementary expression for the determinant of $A(j|i)$. The first step in this direction is similar to the approach in Minc[2,pp.147–149] who considered the more general case of establishing a general expression for the submatrix $A(j|i)$ results from the tridiagonal matrix. Since the submatrix $A(j|i)$ results from the tridiagonal matrix A after the deletion of the row j and the column i, the submatrix is in the lower triangular block form if $i < j$, in diagonal block form if $i = j$, and in upper triangular block form if $i > j$. Each of these "block" is a square submatrix of A. the determinant of such block matrices if the product of the determinant of the diagonal blocks. As consequence, we obtain

$$det\, A(j|i) = det\,(A[1,...,i-1]).det\,(A[i,...,j-1|i+1,...,j]).det\,(A[j+1,...,d])$$

if $1 \le i < j \le d = n - 1$,

$$det\,A(j|i) = det\,(A[1,...,i-1]).det\,(A[j+1,...,d])$$

if $1 \le i < j \le d$

$det\,A(j|i) = det\,(A[1,...,j-1]).det\,(A[j+1,...,i|j,...,i-1]).det\,(A[i+1,...,d])$
if $1 \le i < j \le d$ As a convention, if u>v then $det\,(A[u,...,u]) = 1$.

The final step towards an elementary expression of $det\ A(j|i)$ requires the determination of the determinants of the diagonal block matrices. An elementary expression for the matrices of the type $A[1,...,l]$ can be taken directly from lemma 2.1. Since the structure of the matrices of the type $A[l+1,...,d]$ is identical to the structure of the matrices of the type $A[1,...,d-l]$, Lemma 2.1 also leads to an elementary expression for the determinants of these matrices – we only take into account that the indices have the offset l. Consequently, we obtain

$$det\ A[l+1,...,d] = \sum_{k=0}^{d-1}\left(\prod_{u=1}^{d-k} p_u\right)\left(\prod_{v=d-k+1}^{d+1} q_v\right)$$

If $1 \le i < j \le d$ then matrix $(A[i,...,j-1|i+1,...,j])$ reduce to a lower triangular matrix. Similarly, if $1 \le i < j \le d$ then matrix $A[j+1,...,i|j,...,i-1]$ is upper triangular. It follows that

$$det\,A[i,...,j-1|i+1,...,j] = (-1)^{j-i}\prod_{k=1}^{i-1} q_k (1 \le i < j \le d)$$

and

$$det\,A[j+1,...,i|j,...,i-1] = (-1)^{j-i}\prod_{k=j+1}^{i} p_k (1 \le i < j \le d)$$

Let B : (n-1)x(n-1) be the fundamental matrix of the general random walk with absorbing boundaries at state 0 and n. the entries b_{ij} of the matrix B are

$$b_{ij} = \frac{\left[\sum_{k=0}^{i-1}\left(\prod_{u=1}^{i-k-1} p_u\right)\left(\prod_{v=i-k}^{i-1} q_v\right)\right] * \left[\prod_{k=i}^{j-1} q_k\right] * \left[\sum_{k=0}^{n-j-1}\left(\prod_{u=j+1}^{n-k-1} p_u\right)\left(\prod_{v=n-k}^{n-1} q_v\right)\right]}{\sum_{k=0}^{n-1}\left(\prod_{u=1}^{n-k-1} p_u\right)\left(\prod_{v=n-k}^{n-1} q_v\right)}$$

for $1 \leq i \leq j \leq n-1$ and

$$b_{ij} = \frac{\left[\sum_{k=0}^{j-1}\left(\prod_{u=1}^{j-k-1} p_u\right)\left(\prod_{v=j-k}^{j-1} q_v\right)\right] * \left[\prod_{k=i+1}^{j} p_k\right] * \left[\sum_{k=0}^{n-j-1}\left(\prod_{u=i+1}^{n-k-1} p_u\right)\left(\prod_{v=n-k}^{n-1} q_v\right)\right]}{\sum_{k=0}^{n-1}\left(\prod_{u=1}^{n-k-1} p_u\right)\left(\prod_{v=n-k}^{n-1} q_v\right)}$$

for $n-1 \geq i \geq 1$.

CHAPTER 4

RESULTS AND ANALYSIS

4.1 OVERVIEW

This chapter presents a discussion on the results of the study. It also assesses how far the objectives of the study have been achieved. The summary of data and computation of procedure will also be displayed.

Data was obtained from student's records of Kwabenya Atimonic Cluster of Junior High Schools student assessment sheet (Cumulative Records). Historical data from 2010 to 2012 of students from Kwabenya Atomic Cluster of Junior High Schools will be investigated. Individual student's record containing academic status and grade reports of a random sample of 1200 students will be examined. These are students who either graduated from or dropped out of the School.

A student who stayed in the school only for one term provided one piece of transition data from the state F (completed first term) to LS (Leave school without permission) or to DO (dropped out) after that term. A student who stayed two or more terms provided one piece of transition data for each term that he / she enrolled or graduated from or dropped out of the school. A transition from one state to another is considered to take place at the end of the term.

For the purpose of this paper, a Markov chain of 8 states is identified. A student in Kwabenya Atomic Cluster of Junior High Schools should complete 9 terms to graduate. Each state is classified as follows:

F: First year, a student who successfully scored less than 300 marks. S: Second year, a student who successfully scored between 150 – 300 marks Se: Senior, a student who successfully scored more than 300 marks.

G: Graduated from the school.

DO: Dropped out from the school.

LS: Leave school without permission, perhaps to take care of personal problems.

T: Transferred between Kwabenya Atomic Cluster of Junior High Schools to other schools in the Municipality.

NP: Not promoted, Students who might have not scored the required marks need but through parents and teachers advice were asked to go the next class.

After a term is finished, a first year student, say, who successfully completed first term provided one piece of transition data - from (F) to (F). A frequency transition data would be drawn and the each row total of the frequency table will be used to construct the corresponding probabilities. The matrix of transition probabilities is obtained by dividing each frequency by the appropriate row total. The transition probabilities will be grouped in a canonical form. The results of the study would be tested against the national figure for dropout rate, completion or graduation rate and the average length of time to complete Junior High School.

4.2 SUMMARY OF DATA

	F	S	Se	LS	NP	T	G	DO	TOTAL
F	3392	1072	0	56	0	8	0	64	4592
S	0	3128	1016	32	24	8	0	16	4224
Se	0	0	1008	8	0	0	1008	0	2024
LS	0	0	0	88	0	8	0	0	96
NP	0	0	0	0	24	0	24	0	48
T	0	0	0	0	0	16	16	0	32
G	0	0	0	0	0	0	1	0	1
DO	0	0	0	0	0	0	0	1	1
TOTAL	3392	4200	2024	184	48	40	1049	81	11018

Table 4.1: Frequency Transition Data of 1200 Random Students from Kwabenya Atomic Cluster of Schools

	F	S	Se	LS	NP	T	G	DO
F	0.7387	0.2334	0	0.0122	0	0.0017	0	0.0139
S	0	0.7405	0.2504	0.0076	0.0057	0.0019	0	0.0038
Se	0	0	0.4980	0.0040	0	0	0.4980	0
LS	0	0	0	0.9167	0	0.0833	0	0
NP	0	0	0	0	0.5000	0	0.5000	0
T	0	0	0	0	0	0.5000	0.5000	0
G	0	0	0	0	0	0	1	0
DO	0	0	0	0	0	0	0	1

Table 4.2: Transition Probability Matrix P

4.3 Stages in Analysis of Data and Computation Procedures.

4.3.1 Presentation of data in Canonical Form

Transition matrix of an absorbing Markov chain follows a Canonical form, which means that the transient states come first:

$$\mathbf{P} = \begin{array}{c} \\ \text{TR.} \\ \text{ABS.} \end{array} \begin{array}{c} \begin{array}{cc} \text{TR.} & \text{ABS.} \end{array} \\ \left(\begin{array}{c|c} \mathbf{Q} & \mathbf{R} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right) \end{array}$$

Figure 4.1: Canonical form of the Transition Probability matrix P (After Carroll et al, 2012)

	F	So	Se	NR	NP	T	G	DO
F	0.7387	0.2334	0	0.0122	0	0.0017	0	0.0139
So	0	0.7405	0.2504	0.0076	0.0057	0.0019	0	0.0038
Se	0	0	0.4980	0.0040	0	0	0.4980	0
NR	0	0	0	0.9167	0	0.0833	0	0
NP	0	0	0	0	0.5000	0	0.5000	0
T	0	0	0	0	0	0.5000	0.5000	0
G	0	0	0	0	0	0	1	0
DO	0	0	0	0	0	0	0	1

Table 4.3: Transition Probability Matrix P

.0139
.0038

.5000 0
5000 0.5000 0

$$Q = \begin{matrix} 0.7387 & 0.2334 & 0 & 0.0122 & 0 & 0.0017 & 0 & 0 \\ 0 & 0.7405 & 0.2504 & 0.0076 & 0.0057 & 0.0019 & 0 & 0 \\ 0 & 0 & 0.4980 & 0.0040 & 0 & 0 & 0.4980 & 0 \\ 0 & 0 & 0 & 0.9167 & 0 & 0.0833 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5000 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0. & & \end{matrix}$$

represents the transition probabilities from transition to transition states.

$$R = \begin{matrix} 0 & 0.0139 \\ 0 & 0.0038 \\ 0.4980 & 0 \\ 0 & 0 \\ 0.5000 & 0 \end{matrix}$$

represent the transition probabilities from transient

0.5000 0

state to absorbing states.

I: is an identity matrix, reflecting the fact that we never leave an absorbing state. 0: is a zero matrix, reflecting the fact that it is impossible to move from absorbing state to transition state.

4.3.2 The Fundamental Matrix (B)

Theorem: For an absorbing Markov chain the matrix $I - Q$ has an inverse B and $B = I + Q + Q^2 + \ldots$ For an absorbing Markov chain P, the matrix $B = (I - Q)^{-1}$ is called the fundamental matrix for P. The entry b_{ij} of B gives the expected number of times that the process is in the transient state X_j if it is started in the transient state X_i.

$(I - Q) =$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.7387 & 0.2334 & 0 & 0.0122 & 0 & 0.0017 & 0 & 0.0139 \\ 0 & 0.7405 & 0.2504 & 0.0076 & 0.0057 & 0.0019 & 0 & 0.0038 \\ 0 & 0 & 0.4980 & 0.0040 & 0 & 0 & 0.4980 & 0 \\ 0 & 0 & 0 & 0.9167 & 0 & 0.0833 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5000 & 0 & 0.5000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5000 & 0.5000 & 0 \end{bmatrix}$$

The Fundamental Matrix B is given as $(I - Q)^{-1}$. We need to determine the inverse of the Fundamental Matrix of 6 by 6 matrix. We now used the Matlab to calculate or determine the Fundamental Matrix B.

$$B = \begin{bmatrix} 3.8267 & 3.4429 & 1.6497 & 0.9512 & 0.0391 & 0.1849 \\ 0 & 3.8540 & 1.8467 & 0.4380 & 0.0438 & 0.0876 \\ 0 & 0 & 1.9921 & 0.0945 & 0 & 0.0157 \\ 0 & 0 & 0 & 12.0000 & 0 & 2.0000 \\ 0 & 0 & 0 & 0 & 2.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.0000 \end{bmatrix}$$

4.3.3 The Absorption time (M)

For an absorbing Markov chain, τ is the time of absorption. Let $M_i = E_i\tau$ denote the mean time of absorption starting at state i. Let M stands for the vector whose components are M_i, where $i = 1,2,\ldots,k$. It is known that

$$M = B\varepsilon$$

where ε is the column vector that all of its components are 1.

$$M = \begin{bmatrix} 3.8267 & 3.4429 & 1.6497 & 0.9512 & 0.0391 & 0.1849 \\ 0 & 3.8540 & 1.8467 & 0.4380 & 0.0438 & 0.0876 \\ 0 & 0 & 1.9921 & 0.0945 & 0 & 0.0157 \\ 0 & 0 & 0 & 12.0000 & 0 & 2.0000 \\ 0 & 0 & 0 & 0 & 2.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.0000 \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

1

$$M = \begin{bmatrix} 10.0946 \\ 6.2701 \\ 2.1024 \\ 14.00 \\ 2.00 \\ 2.00 \end{bmatrix} \qquad M(\text{round}) = \begin{bmatrix} 10 \\ 6 \\ 2 \\ 14 \\ 2 \\ 2 \end{bmatrix}$$

Hypothesis Test Results

N	Mean	SE Mean	95% CI	Z	P–value
1200	10	0.12	(9.78,10.23)	8.66	0.0000

4.3.4 Absorption Probabilities (N)

B is a matrix with entries b_{ij} and b_{ij} is the probability that an absorbing chain will be absorbed in the absorbing state X_j if it starts in the transient state X_i.

Then: N = BR

$$\begin{bmatrix} 3.8267 & 3.4429 & 1.6497 & 0.9512 & 0.0391 & 0.1849 \\ 0 & 3.8540 & 1.8467 & 0.4380 & 0.0438 & 0.0876 \\ 0 & 0 & 1.9921 & 0.0945 & 0 & 0.0157 \end{bmatrix} \begin{bmatrix} 0 & 0.0139 \\ 0 & 0.0038 \\ 0.4980 & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 0 & 0 & 0 & 12.0000 & 0 & 2.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.0000 & 0 & 0.5000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.0000 & 0.5000 & 0 \end{bmatrix} * \begin{bmatrix} 0.9336 & 0.0664 \\ 0.9854 & 0.0146 \\ 1 & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Hypothesis Test Results Graduating

Sample p	95% CI	Z	P–value
0.930	(0.8799,0.9800)	4.42	0.0000

Hypothesis Test Results for Dropout

Sample p	95% CI	Z	P–value
0.0400	(0.0015,0.0780)	-0.37	0.708

4.4 Results and Discussion

The transition probabilities for the student flow (Matrix P) may be interpreted as the probability that a student from a given state will be in another state at the next period. It was found that the incoming first year student has a probability 0.7837 of remaining a first year student, a 0.0122 probability of leaving school without permission, a 0.0017 probability of being transferred, probability 0.0139 of dropping out of the system and a 0.2334 probability of advancing to the second year stage. Students are allowed to go through the full academic year of three terms before a decision could be made either not to promote the student or to let he/her progress to the next stage.

There is a 0.7405 probability of remaining a second year student, a probability 0.0076 of leaving school without permission, a probability of 0.0057 of not promoted, a 0.0019 of being transferred, a probability of 0.0038 of dropping out of the system and 0.2504 of advancing to senior stage. The probabilities of remaining at each stage reduce as the students' advances to a higher level. A senior students has 0.4980 probability remaining a third year student.

Students who leave school without permission have 0.9167 of remaining in that state and 0.0833 of being transferred. Students who are transferred to different school within the district have the probability of 0.500 of remaining as transferred students and a 0.500 of graduating. Transfer students are either allowed to continue or move to the next stage or can be asked to repeat the class in their new school. Students who are not promoted have 0.500 probability of remaining as not promoted and 0.500 probability of graduating. Not promoted student may be asked to go to the next stage by intervention from parents or may be asked to repeat the class if they are not able to meet the requirement of moving to another stage.

4.4.1 The Fundamental Matrix B

It follows from the element of the main diagonal of the Fundamental Matric (B) that, a first year student stays for 3.8267 i.e. close to four (4) terms before moving to the next stage. A Second year student needs 3.854 which is also close to four (4) terms before moving to the next stage. The third year student needs an average of about two (2) terms to finish. The delay in moving from the first and the second to the third or the senior stage, which is the stage of graduation, is due to the fact that, students who leave school without permission comes in and out of the system at any time to join the class, these students may not be promoted or may be asked to repeat a class if they are not able to meet the requirement from moving from one stage to the other. Headteachers cannot sack such students because of the Free Compulsory Universal Basic Educational (FCUBE) Policy – All Ghanaian peoples of school going age must be in a school. At the senior or the third year stage, students spent the required length of time of two (2) terms at that stage. This may be attributed to the fact that, Students cannot be stopped from graduating once they finish registering with West African Examination Council (WAEC) eventhough students move and out of the system. Student at this level turn out to be truant. It could be observed again that, the students leave school without permission takes an average of about 12 terms to get to the senior stage and once they get there, they would need only two (2) terms to finish as required. This may be due to the fact that, they repeat class more than once. They may be asked to repeat a class and if they are not able to meet the requirement to move to the next stage, they can again be asked to repeat the same class again.

4.4.2 Absorption Time Matrix (M)

We see from the vector t that, a freshman stays on an average of ten (10) terms in the system before reaching one of the absorbing stages, i.e. the graduating and the dropping out state. A second year student needs an average of about six (6) terms before reaching one of the absorbing stages. A third year student needs two (2) terms before reaching one of the absorbing stages. Students at this stage need not to go through the normal three terms before graduating because after the West African Examinations on June, ends the second term and get to the absorbing stage. Students who leave school without permission need an average of about 14 terms before reaching one of the absorbing stages.

Based on the p-value of 0.0000, we reject H_0 and conclude that number of terms required by first year students of ten (10) in this study is significantly different from the standard or normal nine terms. Thus, some students uses more than nine terms to complete Junior High School.

4.4.3 Absorption Probabilities Matrix N

Looking at the matrix N, we notice that, a first year student has the probability of 0.9336 of progressing to graduating stage and a probability of 0.07 of moving to dropping stage. We notice that the probability of graduating increase as the students progress to the higher level. A third year student has a probability of one (1) of progressing to graduation stage. This implies that, there is a certainty that a student would progress to graduating stage once he/she get to the senior stage whether student leave school without permission, not promoted or transferred. Leave school without permission, Not Promoted and Transfer students have probability of one (1) of progressing to graduation stage.

From the p–value of 0.0000, we reject H_0 and conclude that the completion rate of (0.93) among students in the Kwabenya Atomic cluster of schools is significantly different from the national completion rate of 0.735. The completion rate at Kwabenya Atomic cluster of schools is higher than the national completion rate of 0.735.

From the p–value of 0.708, we fail to reject the H_0 and conclude that the dropout rate (0.04) among students in the Kwabenya Atomic cluster of schools is not significantly different from the national southern dropout rate of 0.048.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

This chapter presents the conclusions and recommendations of the study.

5.1 CONCLUSIONS

Closed form expressions for the entries of the fundamental matrix of the general random walk with absorbing boundaries have been derived by means of elementary matrix theory. The elementary matrix theory approach is especially useful if the absorption probabilities or the absorption time for a specific state are of interest because few entries of the fundamental matrix must be determined in this case. 'Matlab' was used to determine the all the entries of the fundamental matrix which leads to calculations of the absorption probabilities and absorption time to determine the student flow in an academic institution.

It was found out from the fundamental matrix that, a first year student stays for 3.8267 i.e. close to four (4) terms before moving to the next stage. A Second year student needs 3.854 which is also close to four (4) terms before moving to the next stage. The third year student needs two (2) terms to graduate. A first year student stays on an average of ten (10) terms in the system before graduating or dropping out of school. Second year student stays on an average of six (6) terms before graduating or dropping out of school. A third year student stays for two (2) terms before graduating or dropping out. Not registered student on an average needs on average 14 terms before graduating or dropping out.

It was noticed from the absorption probability matrix (B) that, a first year student has the probability of 0.9336 (93.4%) of progressing to the graduating state. The second years student has the probability of 0.9854 (98.5%) of progressing to the graduating state. We are certain (probability of one (1)) that, once a student get to third year will graduate. We are also sure (with probability of one (1)) that, students who get Transferred, Not registered and Not promoted will graduate with if they continue to be in the system.

Again, the absorption probability matrix reviews that, first year students has the probability of 0.07 (6.6%) of progressing to dropout stage. A second year student has the probability of 0.0146 (1.5%). Third year students do not progress to dropout stage.

5.2 RECOMMENDATIONS

The absorption time matrix shows that students who leave school without permission takes an average of fourteen (14) terms before graduating. This create a lot of inconveniences to headteachers (decision to accept truant students is difficult), teachers (treated topic have to be taught again to build on it to teach other topics) and increases government expenditure since the government have to pay extra capitation grant, exercise books, text books etc.

In order to ensure an effective and efficient management of Basic Education in the District and to reduce the expenses of the government, the following suggestions could be adhered to:

1. An investigation should be conducted by the Ghana Education Office, Abokobi to find out the cause of the problem and find administrative solutions to the problem.
2. I recommend that, Parents and students must be educated on the importance of education and to encourage student the need to go to school regularly.

3. Truant (Leave school without permission) students should be identified and counselled by the Guidance and Counseling Coordinator of the Ghana Education Office, Abokobi to know the effects of their actions. This can help them to perform better to get placement in their selected choice of secondary schools after Basic Education Certificate Exams (BECE)

4. Workshop should be conducted for headteachers to teach them how to keep up to date records to improve such kind of research.

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Appendix A

```
data=[3392 1072 0 56 0 8 0 64 4592 0 3128
1016 32 24 8 0 16 4224
0 0 1008 8 0 0 1008 0 2024
0 0 0 88 0 8 0 0 96
0 0 0 0 24 0 24 0 48
0 0 0 0 0 16 16 0 32
0 0 0 0 0 0 1 0 1
0 0 0 0 0 0 0 1 1
3392 4200 2024 184 48 40 1049 81 11018] [n,m]=size(data); earlyP =

data;

for i=1:n earlyP(i,:) = earlyP(i,:)./earlyP(i,end); end

Transition_Matrix = earlyP(1:n-1,1:n-1)

Q  = Transition_Matrix(1:end-2,1:end-2)

R  = Transition_Matrix(1:end-2,end-1:end)

O = Transition_Matrix(end-1:end,1:end-2)

I_t = Transition_Matrix(end-1:end,end-1:end) disp('Fundamental
Matrix, N = (I - Q)^-1')
```

```
I = eye(size(Q)); N =

inv(I - Q)


disp(' Time to Absorption, t = NC') C =

ones(size(N,1),1);

t = N*C

t_round = round(t)


disp('Asorption Probability, B = NR')

B = N*R
```