KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI

STOCHASTIC MODELING OF CRUDE OIL PRICES

By

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Declaration

I hereby declare that this submission is my own work towards the award of the M. Phil degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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Dedication

I dedicate this thesis to my parents, Mr. and Mrs. Imoro whose unflinching support has brought me this far.

Abstract

The prices of crude oil are largely characterized by shocks due to the flow in supply and demand of oil. In this study, we modeled crude oil prices fluctuations using stochastic differential equations. Analytical and numerical solutions of the three-factor model proposed by Cortazar and Schwartz (2003) are presented based on the current price, the future price and volatility of the crude oil. Our simulations results implementing the model indicated that the simulations achieve better results when as many paths with smaller time interval are used. We also studied the price dynamics of WTI crude oil traded at NYMEX from 2004 to 2014. Our study showed that crude oil prices fluctuate over the years with the highest price recorded in June, 2008 but dropped significantly to $41.12/barrel in December, 2008. We also studied the price dynamics of crude oil futures for the period, April, 2015 to December, 2023 and observed that despite the continuous fall in crude oil prices from November, 2014, futures prices increase continuously. Our simulation results on the value of crude oil options revealed that as the value of crude oil prices increase, the expected value of the option increases.

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Contents

List of Abbreviation

ANN .... Artificial Neutral Network

ARIMA .... Auto Regressive Integrated Moving Average

BDC .... Bulk Distribution Company

BSM .... Black Scholes Merton

CFTC .... Commodity Futures Trading Commission

DM .... Datar–Mathews Method

GBM .... Geometric Brownian Motion

GDP .... Gross Domestic Product

IMF .... International Monetary Fund

NYMEX .... New York Merchandise Exchange

OPV .... Oil Price Volatility

SPDs .... State Price Densities

TOCOM .... Tokyo Commodity Exchange

TOR .... Tema Oil Refinery

WTI .... West Texas Intermediate

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CHAPTER 1

INTRODUCTION

1.1 Background of the Study

Crude oil is a naturally occurring unrefined petroleum product composed of hydrocarbon deposits. It can be refined to produce usable products such as oil, diesel and various forms of petrochemicals for industry. Hence, crude oil is regarded as one of the most important commodities in the world (Hubbard, 1998). The prices of crude oil are largely characterized by shocks due to the flow in supply and demand of oil. These shocks bring about high fiscal deficit and high imports bill especially on budgets of nations which are donor supported. For instance Ghana's oil import bill skyrocketed to more than $2billion in 2007, almost three times the bill in 2004 which stood as $775million due to increase in the prices of crude oil in the international market and the high demand of the commodity. It is therefore imperative for Governments to develop possibilities of hedging crude oil prices in the volatile crude oil prices markets. Most Governments have therefore explored various possibilities of hedging crude oil prices in order to ameliorate the dangers associated with the risk of exposure to the variant international crude oil prices. Ghana's jubilee oil field which contains up to 3 billion barrels (480,000,000 m3) of sweet crude oil was discovered in 2007, among the many other oil fields in Ghana. Oil and gas exploration is ongoing in Ghana and the amount of both crude oil and natural gas continues to increase. However the government of Ghana pays heavily in importing crude oil. The Government of Ghana also pays heavily to subsidize the prices of crude oil to consumers in the local market. The Government in 2011 also decided to hedge the sale of its jubilee oil at $107 per barrel to stabilize its budget revenue.

A hedge is an investment position that is intended to offset potential risks that may be incurred by a companion investor. It seeks to provide an insurance against rapid price increases and decreases.

A hedge can be constructed from many types of financial instruments and derivatives such as options, securities futures, and the like. Options are contracts which give the buyer (owner) the right, but not the obligation, to buy or sell an underlying asset at a specified strike price on or before a specified date. Options trading unlike futures and other derivatives gives the trader plenty of extra scope to make leverage bets on the direction of a stock. Options are special tools used to hedge the market volatility of an investment.

Several mathematical approaches have been used to study crude oil price dynamics. These models range from linear models, time series models and stochastic models. In this study we modeled the price dynamics of crude oil as a stochastic process. We also delved into the theory of using options futures to hedge against the risks associated with crude oil price fluctuations. Mathematical models such as the Binomial pricing model, the Heston model and the Black-Scholes-Merton model can be used to price options. This research used the Black-Scholes-Merton pricing model. This was used to determine the value of an option at an expiring date. This will go a long way to assist Governments and Stakeholders in the crude oil business, to adjust or hedge the volatile crude oil prices in the international market.

1.2 Problem Statement

Oil is the most traded commodity, with world exports averaging US $1.8 trillion annually between 2007 and 2009, which amounted to about 10 percent of total world exports in that period (IMF, 2011).

According to Whipple (2012), the International Monetary Fund (IMF) has estimated that the demand of oil has been growing at a rate of circa 800,000 to one million barrels per day, in recent years, all though some foresee this rate of increase declining. The IMF has also predicted Global oil Production dropping by one per cent (1%) annually when the decline comes. Should this occur, oil prices will jump by 60%, and can therefore cause economic instability in many nations.

It is observed that the rise of crude oil prices in the international market greatly affects the demand and supply of the commodity. As supplies decline each year, real oil prices would continue to rise until demand destruction caused by unfavorable oil price brings supply and demand back into a balance. Hence changes in oil market conditions have direct and indirect effects on the global economy, including on growth, inflation, external balances, and poverty.

Many developed countries continue to prosper through exploitation of crude oil from the developing countries. This is because the developing countries lack the efficient knowhow and cash flow to exploit this valuable natural resource (Papapetrou, 2001).

Despite joining the league of oil producing nations in 2010, the government of Ghana spends heavily in importing crude oil for processing and use. For instance in 2013, Ghana spent $2.6 billion to import finished petroleum products from Europe for local consumption. This can be attributed to the fact that Ghana's only refinery, Tema Oil Refinery (TOR), is saddled with debts and cannot refine enough oil to supply the country's energy needs. Ghana's government also spends an average of US $432 million yearly on fuel subsidies only.

Crude oil price volatility undoubtedly poses a threat to the development of every aspect of the economy. It is therefore imperative for governments to adopt measures to hedge crude oil price risks in order to insulate their citizens against the negative effects of these price changes. This research therefore seeks to demonstrate the use of options futures as a means of hedging the underlying price volatilities in the international crude oil market through stochastic modeling.

1.3 Objectives of the Study

With the impact of crude oil prices on a nation's economy, it is prudent to study the dynamics of crude oil price fluctuations and also imperative for governments to develop strategies to hedge these prices in order to minimize the impact of the risk of exposure to international crude oil prices on the economy.
The objectives of this research are:

1. To model crude oil prices using stochastic differential equations.
2. Use numerical simulations to implement the Cortazar and Schwartz (2003) three-factor model for crude oil futures.
3. Use the Black-Scholes-Merton pricing model to determine the value of an option at an expiring date.

1.4 Methodology

This research employed the use of stochastic differential equations to model the prices dynamics of crude oil for a ten year period (2004 – 2014). Much emphasis was placed on the three-factor model developed by Cortazar and Schwartz (2003). Analytical solutions of the model as well as numerical simulations were presented to study this model effectively. The research also underscored basic hedging strategies using options and futures. The Black-Scholes-Merton (BSM) model was used to determine the payoff value of crude oil options. Unlike other options pricing models, the B-S-M model allows for the estimate of the value of any option using a small number of inputs and has been shown to be remarkably robust in valuing many listed options. In this study, we used the implicit Crank-Nicolson method to solve the PDE. Numerical simulations are performed using MATLAB software.

Data for the research were taken from monthly and annual West Texas Intermediate (WTI) crude oil prices traded at the New York Merchandise Exchange (NYMEX) for the period 2004 to 2014.

1.5 Justification of the Study

Crude oil price fluctuations and its inherent effects on the economies of nations are unavoidable since these prices are moderated and influenced by the international market. Kuncoro (2011) established that the volatility of oil prices causes the prices of metal, food grains and other commodities to go up sharply which has high political implications. He further indicated that price fluctuations will increase a household's income risk and a potential output loss for businesses and increase government subsidies.

International price shocks have over the years presented some unfortunate challenges to the Ghanaian economy by way of directing governments resources from social interventions towards subsidies. It at a point in time caused shortage and panic in the system. The most recent one came in June, 2014 as the Government of Ghana tried to take off the gap created between international prices and domestic prices, thereby creating a misunderstanding in the process between government and the Bulk Distribution Companies (BDC's). Another impact of shock was felt in 2003 when a substantial percentage of GDP was spent on subsidies.

However, the Ghanaian government has over the years exhibited a great zeal of hedging to offset the impact of oil price volatilities on her economy. For instance, in March 2010, the Ghanaian parliament approved a petroleum revenue management bill which committed the country of saving a minimum of 30% of its oil revenues in ‘heritage’ and ‘stabilization’ funds to be reinvested in the country’s oil wealth for future generations, and to smooth out the impact of oil price fluctuations on the economy. Also, in October 2010, Ghana decided to hedge its share of production from the jubilee field with put options to secure a stable minimum price of oil produced. The country also embarked on an import hedging program by buying call options from several international banks to protect her economy and citizens from the risk of rising global oil prices. Again, in 2011, as part of a petroleum price risk program, the Government of Ghana decided to hedge 50% of crude oil in order to insulate consumers from hikes of crude oil prices in the world market. In 2011 due to fears in a drop in oil prices, which had the potential of dislocating the economy, the Ghanaian Government was compelled to hedge her crude oil exports at $107 per barrel. It was projected that Ghana would earn approximately $584 million from oil exports representing about 1.9% of Gross Domestic Product (GDP). From September, 2014, the Ghanaian Government also decided to introduce quarterly hedging program as a measure to avoid huge debt accumulating from subsidizing of price differentials on petroleum products. (Source; Ministry of Finance and Economic Planning)

This research therefore seeks to model crude oil price fluctuations as a stochastic process and also to demonstrate how prudent and financially beneficial it is to go into hedging using futures options. Options are more flexible compared to other financial derivatives that are used in price risk management. The research further demonstrated how the values of options are determined at an expired date using the Black-Scholes-Merton options pricing model. The Black-Scholes-Merton model is used to price European options. It enables the calculation of very large number of options in a very short time. This research further provides an insight on the application of mathematical modeling in industry and finance.

1.6 Organisation of the Thesis

The thesis consists of five chapters. Chapter one deals comprehensively with the introduction comprising the background to the study, problem statement, objectives of the study as well as the justification of the research and thesis organization. Chapter two will extensively deal with the review of literature which is relevant to the study. Chapter three discusses the mathematical models as well as the data that are used for the research. Chapter four discusses the numerical methods as well as the computer software that are used in analyzing the data of the study. In chapter five we will discuss the findings of the research and then make the relevant conclusions and recommendations related to the research.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

In the last few decades, several researchers have used different mathematical models to study the price dynamics of crude oil. Other researchers have also developed different crude oil pricing models that have been used to demonstrate how options and other financial derivatives are employed in hedging crude oil price risks in the Global market. In this chapter, some of these literature are reviewed, particularly the ones that have a bearing with the objectives of this study.

2.2 Stochastic Modeling and Hedging of Crude Oil Prices

Mark (2005) explained that recent studies of crude oil price formation emphasize the role of interest rates and convenience yield (the adjusted spot-futures spread), confirming that spot prices mean- revert and normally exceed discounted futures. He further asserts that these studies do not explain why such "backwardation" is normal. Also, models derived in these studies typically explain only about 1 percent of daily returns, suggesting other factors are important, too. The author specified a structural oil-market model that links returns to convenience yield, inventory news, and revisions of expected production cost. Although it's predictive power is only a marginal improvement, his model fits the data far better. In addition, he found reversion of spot to futures prices only when backwardation was severe. His results show that convenience yield behaves nonlinearly, but price response to convenience yield is also nonlinear. He concluded that, futures are informative about future spot prices only when spot prices substantially exceed futures.

According to Krul (2008), futures contracts depend on, when considering deterministic interest rates, the spot price St and the convenience yield δt. The former he said is assumed to follow a Geometric Brownian Motion (GBM) and the latter is usually calibrated via market data every two days, using the futures contracts. The market shows however that the convenience yield behaves stochastically and has a mean-reverting property. He further explained that convenience yield follows an Ornstein-Uhlenbeck process driven by Brownian motion. He calculated both processes are using the Kalman filter method. His results showed that convenience yield can became negative which can results in cost of carry arbitrage possibilities. He therefore introduced the Cox-Ingersoll-Ross process for the dynamics of the convenience yield. The author found strong evidence for the adequacy of his model. He further explained and tested the extended Kalman filter for the Ornstein-Uhlenbeck process, as well as for pricing put options on these futures contracts.

Hosseini (2007) considered the two and three factor modeling of oil futures prices under the risk neutral measure as well as the volatility term structure of futures returns. They analyzed the two and three-factor modeling of oil futures prices developed by Schwartz (1997) and Cortazar and Schwartz (2003) and used mathematical and financial definitions to derive analytical solutions to futures contracts on the commodity.

Tran (2010) considered three different commodity models that take into account the mean reverting nature of commodity prices. The author studied these three models based on two aspects: discretization and filtering. For the first study, he observed and compared theoretically and empirically two well-known discretization schemes, namely, the Euler scheme and the Milstein scheme. His study turns out to be useful for the filtering aspect. Indeed, once a model has been put in a state space form which can be obtained by using a discretization technique, then this enables filtering techniques to be applied to solve the filtering recursion problem for the model. The second study aims to observe and compare the performance of the three well-known filtering techniques, namely, the Kalman filter, the Extended Kalman filter and the Particle filter. He implemented these three filters for the second and third models using MATLAB. The data he utilised to test the models involved futures contracts, since in most commodity markets the futures price is more flexible and easily observed than the spot price of a commodity.

Using a unique, hand-collected data set on hedging activities of 150 U.S. oil and gas producers, Mohamed, et al. (2013) studied the determinants of hedging strategy choice. They also examined the economic effects of hedging strategy on a firm's risk, value and performance. They modeled the hedging strategy choice as a multi-state process and used several dynamic discrete choice frameworks with random effects to mitigate the unobserved individual heterogeneity problem and the state dependence phenomena. Their study presents novel evidence of the real implications of hedging strategy on firm's stock return and volatility sensitivity to oil and gas price fluctuations, along with their accounting and operational performance.

According to Yanbo and Philipe (2006), theories of hedging based on market imperfections imply that hedging should increase the firm's market value. To test this hypothesis, they collected detailed information on the extend of hedging and on the valuation of oil and gas reserves. They examined the hedging activities of 119 U.S and gas producer from 1998 to 2001 and evaluated their effect on firm's value. Even though their study did not use any formidable mathematical model, they however verified that hedging reduces a firm's stock price sensitivity to oil and gas prices.

Also according to Obour (2012), any oil producer or consumer can diversify its risks by transforming its complete dependence on spot oil prices into a variety of exposures to forward, futures and options markets. In the light of these transformations, he analyzed the efficiency of linearly delta hedging with flexos and quantos and further examined their hedging implications. His study fundamentally presents a review of pricing and hedging currency translated options. Currency translated options are options based upon a foreign asset but with a payout that occurs in another currency. His research was intended for Canadian oil producers seeking to mitigate their production and F/X risks.

The adjustment speed of delta hedged options exposure depends on the market realized and implied volatility. Juliusz, et al (2014) observed that by consistently hedging long and short positions in options, one can eventually end up with pure exposure to volatility without any options in the portfolio at all. The results of such arbitrage strategy is based on speed of adjustment of delta hedge option position, more specifically they rely on the interrelation between realized volatility levels calculated for various time intervals. The researchers presented results of simple hedge strategy based on the consistent hedging of a portfolio of options for various worldwide equity indices.

Dempster, et al. (2008), investigated the Valuation and Hedging of spread options on two commodity prices which in a long run are co integrated. They proposed one and two factor models for spot spread processes under both the risk-neutral and market measures. They then developed pricing and hedging formulae for options on spot and futures spreads. To illustrate their results, the authors analyzed two examples of options in the energy markets-the crack spread between heating oil and WTI crude oil as well as the location spread between Brent blend and WTI crude oil.

Chih-Chen, et al (2014) discussed the pricing and hedging of European energy derivatives taken into consideration WTI oil options. Their study extends the mean-reversion dynamic frame work of Pilipovic (1997) and Schwartz (1997). They focused on developing a variety of continuous-time, commodity-pricing and hedging models by analyzing the pricing and hedging errors found in an emperical investigation of options contracts on light sweet crude oil traded on the NYMEX. They concluded that the mean-reversion jump-diffusion and seasonality option-pricing model best describes the extreme price volatility experienced during a financial collapse, but mean-reversion and seasonality option-pricing model offers the best pricing and hedging capabilities for other periods. They revealed that the performances of hedging models are generally consistent with pricing errors.

Roy, et al (2006) studied the hedging problem for European style options on crude oil futures. Local risk-minimizing hedging strategies are derived under the assumption that the dynamics of crude oil futures were described by Merton-type jump diffusion. These were tested empirically using historical data from the NYMEX West Texas Intermediate (WTI) from 2002 - 2007 periods. Their work uses Black-Scholes-Merton delta hedge as a benchmark for comparison and finds out that in crude oil option markets locally risk-minimizing hedges systematically outperform the benchmark by a margin of 13%-18%. Based on this they concluded that locally risk-minimizing strategies are much more robust in hedging crude oil prices than its classical alternatives.

Delphin and Alain (2010) analyzed long-term dynamic hedging strategies relying on term structure models of commodity prices. They proposed a new way to calibrate the models which takes into account the error associated with the hedge ratios. Different strategies, with maturities up to seven years, were tested on the American crude oil futures market. The authors considered three recent and efficient models respectively with one, two, and three factors. The continuity between the models makes it possible to compare their performances which are judged on the basis of the errors associated with a delta hedge. They tested the strategies for their sensitivity to the maturities of the positions and to the frequency of the portfolio rollover. They found that their method gives the better of two seemingly incompatible worlds. They concluded that the three-factor model is by far, the best even if it is more complex.

Brennan and Crew (1997) compared the hedging strategy used by Metallgesellschaft on the crude oil market. They relied on strategies using several term structure models. The authors studied the hedging strategies up to 24 months. Their results showed that those relying on the term structure models are outperformed by that of the German firm by far, all the more as the term structure model is able to correctly replicate the price curve empirically observed.

Martin et al. (2014) investigated the role of volatility and jump risk for the pricing and hedging of derivatives instruments and quantify their associated risk premia in the crude oil futures and option markets. The authors proposed a unified estimation approach that uses both return data and cross section of option prices overtime to consistently estimate parameters, latent variables, and to disentangle the various risk premia. Their estimation results show that jump risk is priced with a significant premium, while no evidence for significant market price of volatility risk exists. Empirical evidence from pricing and hedging exercise confirms these findings.

Veld-Merkoula and de Roon (2003) used a one factor term structure model based on convenience yield. Their goal was to construct hedge strategies that will minimize both spot risk and rollover risk. The researchers also used futures of two different maturities to show that their strategy outperforms the naïve hedging strategy. Naïve hedging strategy is taking a hedging position without taking into consideration the level of hedging required. The optimal hedging position should be such that the expected position from the hedge perfectly offset the underlying risk. They however failed to compare their results with previous work for effective analysis.

2.3 Crude Oil Options Pricing Models

According to Black and Scholes (1973), if options are correctly priced in the market, it should not be possible to make definite profits by creating portfolios of long and short positions in options and their underlying stocks. Using this principle, they derived a theoretical valuation formula for options. This formula is widely known as the Black-Scholes options pricing formula. It is an analytical model that is used as a closed-form solution to price European vanilla options.

Merton (1973) examined the theory of rational option pricing. He deduced a set of restrictions on option pricing formula from the assumption that investors prefer more to less. Since the deduced restrictions are not sufficient to uniquely determine an option pricing formula, he therefore introduced additional assumptions to examine and extend the seminal Black-Scholes explicit formulas for pricing both Call and Put options as well as for warrants and "Down-and-out" options. He further examined the effects of dividend and call provisions on warrant price, and discussed the possibilities for further extensions of the theory to the pricing of co-operate liabilities.

Turner (2010) derived the Black Scholes model of a European option by calculating the expected value of the option. He assumed that the stock price is normally distributed and that the universe is risk-neutral. Using Ito's lemma, the author justified the use of risk-neutral rate in the initial calculations. He finally proofed put-call parity in order to extend European put-options and extend the concept of the Black Scholes formula to value an option with pricing barriers.

Volodymyr and Bardia (2011) discussed the fundamental underlying theory and practice of financial derivative pricing focusing on stock options. They presented the binomial, trinomial and geometric Brownian motion stock price models. They used binomial model to illustrate the main idea of asset pricing theory-no-arbitrage pricing and derived the price of a stock option. Their study also discussed the Black-Scholes-Merton partial differential equation and concluded that the vast majority of derivatives must be priced numerically. The Numerical methods they proposed are: Lattice models, Monte Carlo Simulations, and finite difference methods.

Feng, et al (2011) presented a new spectoral method for solving partial integro-differential equations for pricing European options under the Black-Scholes and Merton jump diffusion models. Their main contributions are:

(i) Using an optional set of orthogonal polynomial bases to yield banded linear systems and to achieve spectral accuracy;

(ii) Using Laguerre functions for the approximations on the semi-infinite domain, to avoid domain truncation; and

(iii) Deriving a rigorous proof of stability for the time discretization of European Put options under both the Black-Scholes and Merton jump diffusion models.

The new method is flexible for handling different boundary conditions and non-smooth initial conditions for various contingent claims.

Nielson (1992) used the risk-adjusted lognormal probabilities to derive the Black-Scholes formula. The author explained the factors N(d_1) and N(d_2). He also showed how the one-period and multi-period binomial option pricing models can be restated so that they involve analogues of N(d_1) and N(d_2) which have the same interpretation as in the Black-Scholes models.

Jigna and Sandeep (2014) underscored the importance of the Black-Scholes-Merton partial differential equation for pricing an option. According to the authors the equation has a very useful application for the trading terminal. They showed that using this model, the trader can find the theoretical value of options (put/call). The researchers also indicated how the B-S-M equation is used to price an option on a variety of assets including securities, commodities, currencies etc. They discussed several methods of solving the B-S-M model and proposed the application of Fourier transformation to determine its solution with due advantages. Their solution provides a fair price of an option (call/put).

Stentoft (2011) considered discrete time GARCH and continuous time SV models and used these for American option pricing. He first of all showed that with a particular choice of framework the parameters of the SV models can be estimated using simple maximum likelihood techniques. Hence the two types of models can be implemented in an internally consistent manner. He then performed a Monte Carlo study to examine their differences in terms of option pricing. He further studied the convergence of the discrete time option prices to their implied continuous time values. The author's results show that there are differences between the two models, though the discrete time GARCH prices converge quickly to the continuous time SV values. Finally, he performed a large scale empirical analysis using individual stock options and options on an index comparing the estimated prices from discrete time models to the corresponding continuous time model prices. His study revealed that, the continuous time SV models do generally perform better than the discrete time GARCH specifications.

Hong-Yi, et al (2008) reviewed the derivations and applications of the "Greek Letters" of options pricing. The "Greek Letters" are defined as the sensitivities of the option price to a single-unit change in the value of either a stable variable or a parameter. The authors introduced the definitions of the "Greek Letters" and provided their derivations for call and put options on both dividends-paying stock and non-dividends stock. They then discussed some applications of the "Greek Letters" and finally showed the relationship between them using the Black-Scholes partial differential equation.

Pann (2001) examined the joint time series of the standard and poor's (S&P) 500 index and near-the-money short-dated option prices with and arbitrage-free model, capturing both stochastic volatility and jumps. His work was based on the jump-risk premia implicit in options. He showed that jump-risk premium uncovered from the joint data respond quickly to market volatility, becoming more prominent during volatile markets. He further indicated that this form of jump-risk premia is important not only in reconciling the dynamics implied by the joint data, but also in explaining the volatility "smirks" of cross-sectional options data.

Matteo and Fernandes (2012) conducted a research which was aimed at estimating the prices of one-month calendar spread call options on crude oil using a stochastic pricing model and Monte-Carlo simulation. They analyzed the very particular commodity market, concentrating their efforts on the description of energy markets. Their research also pays great attention to the idiosyncratic characteristics of commodity prices with particular emphasis on crude oil, and the underlying commodity of the calendar of spread options pricing. They concluded that low values for the state variables volatilities and correlations seem to be the drivers of the underestimations of options prices after they performed sensitivity analysis and computed the implied values of some pricing parameters.

Siddhivinayak and Imad (2009) presented a model based on multilayer feed forward Neutral Network to forecast crude oil spot prices direction in the short-term. Their study dwells on finding an optimal ANN model structure. Their approach is to create a benchmark based on lagged value of pre-processed spot price, then preprocess futures price for the short-term. They concluded their study that future prices of crude oil do not hold new information on spot price direction.

Dontwi, et al (2010), studied the applications of options in hedging crude oil price risks. Their study reveals how options can be used for hedging crude oil price risks in accordance with a broad-based hedging strategy. They also demonstrated how options are priced using the Black-Scholes model and the benefits of hedging through options. Their study places much emphasis on the potential losses and gains anticipated through hedging. They illustrated this using the scenario in the context of the current oil finds in commercial quantities in Ghana.

Günther (2012) discussed some few assumptions underlying the Black-Scholes model. His aim was to evaluate the model and to see whether it works well in the real market. Hedging simulations were carried out in his study for both European and digital call options. The simulations were based on Monte-Carlo simulations of an underlying stock. The author placed emphasis on delta hedging, which appears to work better for digital options. He then concluded that, despite its flaws, the Black-Scholes option pricing model still works for European call options in the real market whiles hedging digital call options is in general is difficult.

Noureddine (2005) recognized both the consumption, an investment aspect of crude oil and proposed a levy process for modeling uncertainly and options pricing. He did calibration to crude oil futures options which show high volatility of oil futures prices, fat-tailed and right-skewed market expectations, implying a higher probability mass on crude oil prices remaining above the futures level. These findings support the view that demand for futures contracts by investors could lead to excessively high price volatility.

In order to find out what derives the price of crude oil, Bogdana (2013) discussed the Geometric Brownian Motion (GBM) and Mean-Reversion Model which are widely used for other commodities and returns modeling. He extended these stochastic modeling approaches with factors describing macro-economic conditions through oil price volatility channels modeled within GARCH framework, in order to capture the expectations and impact on oil prices. The study reveals that simple stochastic models for oil prices demonstrate that drift estimations are very uncertain but are more reliable for the GBM model. The main finding of the study is that crude oil price process has a drift, but it changes once in a while and it may be assumed to be constant but shorter than sample time periods.

According to Carmona and Durrleman (2003), spread options are ubiquitous in the financial markets, whether they are equity, fixed income, foreign exchange, commodities, or energy market. They presented a general overview of the common features of all spread options by discussing in details their roles as speculation devices and risk management tools. Their study describes the mathematical framework used to model spread options. They also reviewed the numerical algorithms used to actually price and hedge spread options. Their research further reveals that, despite an extensive literature on the pricing of spread options in the equity and fixed income market, information about the various numerical procedures which can be used to price and hedge them on physical commodities is more difficult to find. The authors however made a systematic effort to choose examples from the energy market in order to illustrate the numeric challenges associated with these options. They reviewed the two major avenues to modeling energy price dynamics, and explained how pricing and hedging algorithms can be implemented both in the framework of models for spot price dynamics as well as forward curves dynamics.

2.4 Crude Oil Price Volatility Models

To analyze the volatility structure of commodity derivatives market, Carl, et al (2012) assumed a generalized hump-shaped volatility specification that entails a finite - dimensional affine model for the commodity futures curve and quasi-analytical prices for options on commodity futures. An empirical study of the crude oil futures volatility structure was carried out using an extensive data base of options prices as well as futures prices spanning 21 years. Their study concluded that factor hedging depicts that the hump-shaped feature is more pronounced when the market is volatile.

Walid, et al (2014), indicated that both the long memory and asymmetric behavior characterize the conditional volatility of oil and stock market returns. The authors used the DCC - FIAPARCH model to examine the time-varying properties of conditional return and volatility of crude oil and U.S stock markets as well as their dynamic correlations (DCC) over the periods 1988 - 2013. Their study shows that DCC - FIAPARCH model explicitly accounts for long memory and asymmetric volatility effects enabling investors to effectively hedge the risk of their stock portfolios with lower cost, as compared to the standard DCC - GARCH model.

Lubna and Ajith (2013) underscored the vital role oil plays in the global economy. According to the authors oil is an important source of energy representing an indispensable raw material and as a major component in many manufacturing processes and transportation. They consented that oil price suffer from high volatility and fluctuations. In global markets, it is the most active and heavily traded commodity. The authors reviewed various studies that emerged to discuss the problem of predicting oil prices and seeking access to the best outcomes. A comprehensive survey covering the previous methods and some results and experiments were presented in their work with a focus on and maintaining the necessary steps when predicting oil prices. They however failed to employ a specific mathematical model in their study.

Mark and Andrew (2011) examined the relationship between the price of oil and the position of various classes of traders in crude oil futures and options. They used position data from the commitment of traders report, published weekly by the Commodity Futures Trading Commission (CFTC). Their study reveals that a statistically significant correlation is evident between changes in position held by “money managers” and the price of oil. “Money managers” is a category of speculators that includes hedge funds. This statistical relationship is weaker for other classes of speculators and for commercial hedgers.

Study on the crude oil import demand behavior in Ghana by Marbuah

(2013), brings to the fore an understanding of the key drivers of crude oil imports demand using the Autoregressive Distributed Lag modeling framework (ARDL). He estimated variant short-run and long-run import demand models for crude oil using time series data over the period 1980-2012. His results show that crude oil import is the real effective exchange rate, domestic crude oil production and population growth. The author revealed that real economic activity is the most robust and dominant driver of crude oil demand with mixed estimates of inelastic and elastic co-efficient in the short-run and long-run, respectively.

Dependency on oil-derived fuels in various sectors, most notably in mobility has left the global economy vulnerable to several macroeconomic economic side effects. In this light, Zohra, et al (2014) reviewed the interactions between global macroeconomic performance and oil price volatility (OPV). The authors explained that oil price volatility is intrinsic in commodity markets, but has been advancing at a faster rate in the crude oil market in comparison to other commodities over the past decade, reflecting the status of oil as the most globalized commodity. Their study shows that OPV has damaging and destabilizing macroeconomic impacts that will present a fundamental barrier to future sustainable economic growth if left unchecked. They recommended a combination of supply and demand-sided policies to ensure macroeconomic isolation from OPV.

Using market prices for crude oil futures options and the prices of their underlying contracts, Ehud, et al (2009) estimated the volatility skew in two ways. The researcher first estimated a cross sectional polynomial structure for each maturity to demonstrate the strength and weakness of a purely mechanical model. He then applied to the empirical data a Merton-Style Diffusion Model, with rich structure on cross sectional constraints on parameters. He tested both the mechanical and diffusion models with respect to their market-to-market accuracy over time, as well as their efficacy and concluded that in hedging option prices change for a long-term.

Ekmekcioglu (2012) in his research on the macro-economic effects of world crude oil price changes highlighted and analyzed the macro-economic advantages of world crude oil prices. He further articulated the oil price changes and economic output which is very imperative in analyzing the business environment in terms of macro-economic factors. His study also plays emphasis on the various aspects of the effects of crude oil price changes in terms of the profitability that they facilitate. The author however failed to employ any mathematical model to study the crude oil price fluctuations.

Figlewski (1989) explained that option valuation models are based on an arbitrage strategy. That is hedging the option against the underlying asset and rebalancing continuously until expiration. This he said is only possible in frictionless markets. The researcher examined the impact of market imperfections and other problems with the 'standard' arbitrage trade, including uncertain volatility, transaction cost, indivisibilities, and rebalancing only at discrete time intervals. He further found that, in an actual market such as that for stock index options, the "standard" arbitrage is exposed to such large risk and transaction costs that it can only establish very wide bounds on equilibrium options prices.

Yang, et al (2002) used a GARCH model to describe the volatility of oil prices in the USA using monthly data from January 1975 to September 2000. Their model describes the determinant of U.S oil prices. Their modeling technique primarily focuses on OPEC production, real U.S GDP, as well as the price and income elasticity of demand for oil in the U.S. They carried out a co-integration test and used an Error Correction Model (ECM) model to investigate the short and long-run relationships between oil demand and oil price, real GDP and natural Gas and coal prices in order to determine the price and income elasticity of demand. They then performed a simulation of potential oil prices under different scenarios of reduction in OPEC and concluded that OPEC production reduction will result in increases in oil prices, but the magnitude and duration of the increase depends on the size of the OPEC reduction and increase of domestic production by the U.S or other non OPEC producers.

Samii (1992) used the cost of carry model to examine WTI crude oil futures prices. He used daily data from September 20, 1991 to July 15, 1992 and monthly data from January 1984 to June 1992. His results suggest that interest rates do not have a clear influence on oil prices. However, spot and future prices of oil are highly correlated but the direction of the causal relationship between them is not identified.

Ton and Frederick (2013) showed that three funds are necessary to manage an oil windfall. These are intergenerational, liquidity and investment funds. The researchers emphasized that optional liquidity funds is bigger if the windfall lasts longer and oil price volatility, prudence and the Gross Domestic Product (GDP) share of oil rents are high and productivity growth is high. The authors applied their theory to the windfalls of Norway, Iraq and Ghana. They indicated that the optional size of Ghana's liquidity fund is tiny even with high prudence. They also indicated that Norway's liquidity fund is bigger than Ghana's whilst Iraq's liquidity fund is colossal relative to its intergenerational fund. Their research concluded that only with capital scarcity, part of the oil windfalls should be used for investing. They illustrated how this can speed up the process of development in Ghana despite domestic absorption constraints.

2.5 Crude Oil Price Forecasting Models

Moshiri and Foroutan (2005) modeled and forecasted daily oil price futures, listed in NYMEX, applying ARIMA, and GARCH models, for the period April 1983 – Jan. 2003. they then tested for chaos using embedding dimension, BDS, Lyapunov exponent, and neural networks tests. Finally, they set up a nonlinear and flexible ANN model to forecast the series.. Their results of forecasts comparison among the different models confirm that the ANN model makes better forecasts as the tests for chaos indicate that futures oil price follows a chaotic process.

Ahmad (2011) applied Box-Jenkins modeling approach for the time series analysis of monthly average prices of Oman crude oil taken over a period of ten years. He investigated Basic statistical properties of these series. He did a time series plots which clearly indicated a non-stationary trend which he observed to be first differenced stationary. Sample Auto Correlations (SAC) and Sample Partial Auto Correlations (SPAC) plots were used by the author to make tentative identification of the form and order of Box-Jenkins' Auto Regressive Integrated Moving Average (ARIMA) models. He initially postulated for further analysis several seasonal and non-seasonal ARIMA models. He then estimated and compared their adequacy based on the significance of the parameter estimates, mean square and Modified Box-Pierce (Ljung-Box) Chi-Square statistic. Based on these criterion the researcher recommended a multiplicative seasonal model of the form ARIMA(1,1,5)x(1,1,1) for short term forecasting.

Akomolafe and Danladi (2013) used Box and Jenkins Methodology to forecast crude oil price for 2013. Their aim was to advise Nigeria on the oil price benchmark for her budget using the Auto-regressive [AR (2)] model which they found to be most appropriate. The researchers conducted diagnostic tests which showed that the model was good. Based on the model they further conducted a forecast of crude oil prices for the year 2013, and revealed that the price level will be stable around $100. They recommended a benchmark of $80 per barrel given the nature of the Nigerian economy (being solely oil-dependant) as well as given the expected vagaries in the international price of crude oil.

Wen, et al (2006) proposed a new method for crude oil price forecasting based on support vector machine (SVM). The procedure of developing a support vector machine model for time series forecasting involved data sampling, sample preprocessing, training & learning and out-of-sample forecasting. To evaluate the forecasting ability of SVM, they compared its performance with those of ARIMA and Back propagation Neural Network (BPNN). Their experiment results showed that SVM outperforms the other two methods and is a fairly good candidate for the crude oil price prediction.

Despite their widespread use as predictors of the spot price of oil, oil futures prices tend to be less accurate in the mean-squared prediction error sense than no-change forecasts. This result according to Ron and Lutz (2010) is driven by the variability of the futures price about the spot price, as captured by the oil futures spread. They explained this variability by the marginal convenience yield of oil inventories. Using a two-country, multi-period general equilibrium model of the spot and futures markets for crude oil the researchers showed that increased uncertainty about future oil supply shortfalls under plausible assumptions causes the spread to decline. Increased uncertainty also causes precautionary demand for oil to increase, resulting in an immediate increase in the real spot price. Thus, the negative of the oil futures spread may be viewed as an indicator of fluctuations in the price of crude oil driven by precautionary demand. An empirical analysis of this indicator provides evidence of how shifts in the uncertainty about future oil supply shortfalls affect the real spot price of crude oil.

According to Ellwanger (2014) oil prices are notoriously difficult to forecast and exhibit wild swings or "excess volatility" that are difficult to rationalize by changes in fundamentals alone. He offered an explanation for these phenomena based on time varying disaster probabilities and disaster fears. Using information from crude oil options and futures, the author documented economically large jump tail premia in the crude oil derivative market. He showed that these premia vary substantially over time and significantly forecast crude oil futures and spot returns. His results suggest that oil futures prices overshoot (undershoot) in the presence of upside (downside) tail fears in order to allow for smaller (larger) risk premia thereafter. He further showed that this overshooting (undershooting) is amplified for the spot price because of time varying benefits from holding inventory that work in the same direction. His study concluded that the novel oil price uncertainty measures yield additional insights into the relationship between the oil market and macroeconomic outcomes.

According to Jean-Thomas, et al (2008) empirical research on oil price dynamics for modeling and forecasting purposes has brought forth several unsettled issues. They explained that statistical support is claimed for various models of price paths, yet many of the competing models differ importantly with respect to their fundamental temporal properties. The authors studied this phenomenon using mean-reversion method, with emphasis on forecast performance. They considered three specifications:

(i) Random-walk models with GARCH and normal or student-t innovations,

(ii) Poisson-based jump-diffusion models with GARCH and normal or student-t innovations, and

(iii) Mean-reverting models that allow for uncertainty in equilibrium price and for time-varying convenience yields.

The authors compared forecasts in real time, for 1, 3 and 5 year horizons. For the jump-based models, they relied on numerical methods to approximate forecast errors. Their results based on future price data ranging from 1986 to 2007 strongly suggest that imposing the random walk for oil prices has pronounced costs for out-of-sample forecasting. Their evidence in favor of price reversion to a continuously evolving mean underscores the importance of adequately modeling the convenience yield.

Hung-Chun, et al (2009) assessed the market risk in the international crude oil market from the perspective of VaR analysis. The authors used daily returns of West Texas International (WTI) crude oil prices from December 2003 to December 2007 to indicate that GARCH-SGT model is superior to that of GARCH-T and GARCH-GED models. They revealed that the sophisticated SGT distributional assumption significantly benefits VaR forecasting for WTI crude returns at low and high confidence levels, indicating a need for VaR models that consider fat-tails, leptokurtosis and skewness behaviors. They concluded that the GARCH-SGT model is a robust forecasting approach that can practically be implemented for VaR measurement.

Xuhui (2012) explained that standard asset pricing theory suggests that State Price Densities (SPDs) monotonically decrease with returns. His goal was to ascertain investor beliefs and state price densities in crude oil markets. He estimated that the SPDs implicit in the crude oil market display a time varying U-shape pattern. This he said implies that investors assign high state prices to both negative and positive returns. The author used data of the crude oil market, where speculation and short sales were not regulated, to document how the SPDs are dependent on investor beliefs. He concluded that investors assign higher state prices to negative returns when there are more net short positions, higher dispersion of beliefs in the futures market, and higher demand for out-of-the-money put options.

Layiwola (2014) examined the relationship between WTI crude oil prices and US crude oil inventory using the annual data from 1976 to 2009 using the Structural Dynamic Model. The author estimated linear models using co-integration approach specifically Johansen techniques. He then employed the approach to examine the relationship among WTI crude oil price, crude oil inventory, OPEC crude oil production, OPEC refinery capacity, and employment level and energy intensity. Based on the VAR model, he concluded that the WTI crude oil price receives negative and significant influence from inventory, OPEC production, OPEC refinery capacity and energy intensity; and that employment affects WTI crude oil price insignificantly in positive direction.

Jamal, et al (2014) explained that with the increasing number of quantitative models available to forecast the volatility of crude oil prices, the assessment of the relative performance of competing models becomes a critical task. Their survey of the literature revealed that most studies tend to use several performance criteria to evaluate the performance of competing forecasting models; however, models are compared to each other using a single criterion at a time, which often leads to different rankings for different criteria—A situation where one cannot make an informed decision as to which model performs best when taking all criteria into account. In order to overcome this methodological problem, the authors proposed a multidimensional framework based on an input-oriented radial super-efficiency Data Envelopment Analysis (DEA) model to rank order competing forecasting models of crude oil prices' volatility. However, their approach suffers from a number of issues. In this paper, we overcome such issues by proposing an alternative framework.

Manescu and Robays (2014) demonstrated how the real-time forecasting accuracy of different Brent oil price forecast models changes over time. They found considerable instability in the performance of all models evaluated. Based on this they argued that relying on average forecasting statistics might hide important information on a model‘s forecasting properties. The researchers proposed a four-model combination (consisting of futures, risk-adjusted futures, a Bayesian VAR and a DGSE model of the oil market) that predicts Brent oil prices more accurately than the futures and the random walk up to 11 quarters ahead, on average, and generates a forecast whose performance is remarkably robust over time. Also their model combination reduces the forecast bias and predicts the direction of the oil price changes more accurately than both benchmarks.

Chia-Lin, C., et al (2010) examined the performance of four multivariate volatility models, namely CCC, VARMA-GARCH, DCC and BEKK, for the crude oil spot and futures returns of two major benchmark international crude oil markets, Brent and WTI. They calculated optimal portfolio weights and optimal hedge ratios, and suggested a crude oil hedge strategy. Their results show that the optimal portfolio weights of all multivariate volatility models for Brent suggest holding futures in larger proportions than spot. For WTI, however, DCC and BEKK suggest holding crude oil futures to spot, but CCC and VARMA-GARCH suggest holding crude oil spot to futures. Also, the calculated optimal hedge ratios (OHRs) from each multivariate conditional volatility model give the time-varying hedge ratios, and recommend to short in crude oil futures with a high proportion of one dollar long in crude oil spot. The authors concluded that the hedging effectiveness indicated that DCC (BEKK) is the best (worst) model for OHR calculation in terms of reducing the variance of the portfolio.

Bakanova (2011) evaluated different procedures for modeling and forecasting crude oil price volatilities. He examined the relative accuracy of these forecasts using data from the light, sweet crude oil futures market traded at New York Merchantile Exchange (NYMEX). The researcher employed a range-based volatility estimators and the model-free methodology to extract implied volatility from prices of options on crude oil futures contract. His work shows that the model-free implied volatility, although biased, has the predictive power and is an efficient measure of future realized volatility of oil prices in the international market.

CHAPTER 3

METHODOLOGY

3.1 Introduction to Crude Oil Modeling

In the last few decades, crude oil prices have presented large variations-they have steadily risen from about 25dollars a barrel to over 130dollars a barrel in May, 2008, and then dramatically dropped during the crises of 2008 (Tatyana, 2010). These movements influence capital budgeting plans as well as the value of foreign-dominated asset investments. Crude oil price volatility could also bring a lot of economic instability in all oil exporting and oil importing countries, in both developed and developing countries. Oil price shocks have been cited as causing adverse macro-economic impacts on aggregate output, price, and employment across the world. The movements in the prices are very complex, and unpredictable. Hence, predicting its future price and managing the risks associated with its prices, is very crucial for governments and businesses.

Oil price modeling is, therefore very vital to agents and policy makers in the oil market. There have been many efforts to exploit models that could explain the behavior of crude oil prices and forecast it accurately in spot and exchange trade markets. These models include linear structure models, linear and non-linear time series models and stochastic models

The traditional linear structure models for forecasting crude oil prices have not been very promising particularly in the case of complex series such as oil prices. Although the linear and nonlinear time series models have done a better job in forecasting oil prices, there is yet room for improvement. If the data generating process is non-linear, applying linear models could result in large forecast errors. Model specifications in non-linear modeling can also be very case-dependent and time-consuming (Saeed and Faezeh, 2004).

The stochastic models have been proven to be robust in explaining the demand and supply movements as well as useful in forecasting oil prices. In this study, the stochastic modeling technique is used to model crude oil futures prices. Stochastic modeling is a technique of presenting data or predicting outcomes that take into account a certain degree of randomness, or unpredictability. It concerns the use of probability to model real-world situations in which uncertainty is present.

The use of stochastic model reflects a pragmatic decision on the part of the modeler that such a model represents the best currently available description of the phenomenon under consideration, given the data that is available and the universe of the models known to the modeler.

The daily, weekly, monthly and yearly prices of crude oil traded on NYMEX for WTI are analysed for the period 2004-2014. The research also proposes a hedging technique using options to minimize the risks associated with crude oil price fluctuations.

3.2 Stochastic Characteristics of Crude Oil Prices

3.2.1 Mean Reversion

A mean reversion process refers to a situation where prices do not grow indefinitely. In the short run fluctuations might occur, but in the long run prices should revert towards their marginal cost of production (Dixit, et al, 1994). Mean reversion is primarily premised on the assumption that the logarithm of the oil price reverts to its long-term mean.

3.2.2 Convenience yield

Convenience yield according to Brennan and Schwartz (1985) is defined as "the flow of services that accrues to the owner of a contract for future delivery of a commodity". Since 1939, the convenience yield plays a crucial role in the explanation of the relationship between spot and futures prices in commodity markets. It indeed appears as a way to explain backwardation, a situation where the futures price is lower than the spot price (Delphine, 2009). Significant convenience yields in the WTI futures markets have been presented by Gibson and Schwartz (1990). Also, there is a growing realization that convenience yields are stochastic and seasonal. Milonas and Thomadakis (1997) have modeled convenience yields as call options written on a futures contract with expiration time some intermediate period prior to maturity and striking price, the maximum price that intermediate futures can take given the expected available supplies then. Nikolaos and Thomas (2001) explain that if convenience yields are part of the observed futures prices in both Brent and WTI futures markets, their price spread will be due to the relative changes in the two convenience yields, ceteris paribus.

3.2.3 Jumps and spikes

Since 2004, crude oil has experienced significant volatility in prices caused mainly by high uncertainties driven both by supply and demand side factors, geopolitical considerations, and speculation. Crude oil prices rose from 2004 to historic highs in mid-2008, only to fall precipitously in the last four months of 2008, shedding all the gains of the preceding four and a half years. The steep price increase experienced from January 2007 to July 2008, in particular, was challenging for many non-oil producing developing countries. While the sharp drop in prices since August 2008 was welcomed news for consumers. The World Bank in 2008 reported that, the extent of pass-through of the rise in world oil prices to the domestic market showed that developing countries could not keep up with the price increases between January 2007 and August 2008. Correspondingly, retail prices in developing countries increased less than in developed countries during the period (source: Institute for Fiscal Studies-Ghana, 2015).

On December 23, 2008, WTI crude oil spot price fell to US $30.28 a barrel, the lowest since the financial crises of 2007-2010 began. The price sharply rebounded after the crisis and rose to US $82 a barrel in 2009. On 31 January 2011, the Brent price hit $100 a barrel for the first time since October 2008, on concerns about the political in Egypt. For about three and half years the price largely remained in the 90˘120 range. In the middle of 2014, price started declining due to a significant increase in oil production in USA, and declining demand in the emerging countries. By 12 December 2014 the price of benchmark crude oil, both Brent and WTI reached their lowest prices since 2009. Brent crude oil dropped to US $62.75 a barrel for January delivery on the London-based ICE Futures Europe exchange and futures for WTI for January settlement slid to $58.80 a barrel in electronic trading on the NYMEX. This represents a 40 per cent decrease in 2014 (OPEC, 2015).

3.3 Definition of Some Mathematical Terms under Stochastic Modeling

3.3.1 Definition of stochastic process

A stochastic process is a statistical process involving a number of random variables depending on a variable parameter (which is usually time).
A stochastic process $\{X_t : t \geq 0\}$ is said to be:

a. Stationary if $\forall\ t_1 < t_2 < ... < t_n$ and $h > 0$, the random n-vectors

$\{X_{t_1}, X_{t_2}, ..., X_{t_n}\}$ and $\{X_{t_{1+k}}, X_{t_{2+k}}, ..., X_{t_{n+k}}\}$ are identically distributed, that is time shifts leave probabilities unchanged.

b. Gaussian if $\forall\ t_1 < t_2 < ... < t_n$, the n-vector $\{X_{t_1}, X_{t_2}, ..., X_{t_{n-1}}\}$, is multivariate normally distributed.

c. Markovian if $\forall\ t_1 < t_2 < ... < t_n$,
$P(X_{t_n} \leq x | X_{t_1}, X_{t_2}, ..., X_{t_{n-1}}) = P(X_{t_n} \leq x | X_{t_{n-1}})$ that is, the future is determined by the present and not the past.

3.3.2 The Wiener process

The Wiener process also called Brownian motion is a Markov stochastic process. A Markov stochastic process is a particular type of stochastic process where only the current value of a variable is relevant for predicting the future movement.

Consider a simple random walk; $\{X_n\}_{n \in N}$ on the lattice of integers, $\mathbb{Z}$,

$$X_n = \sum_{k=1}^{n} \varepsilon_k$$

Where $\{\varepsilon_k\}_{k \in N}$ is a collection of independent, identically distributed random variables with $P(\varepsilon_k = \pm) = \frac{1}{2}$

From the central limit theorem,
$\frac{X_N}{\sqrt{N}} \to N(0, 1)$, i.e. Gaussian variable with mean 0 and variance 1.
In the distribution, as $N \to \infty$, this defines the piece-wise constant random function:

$$W_t^N = \frac{X_{\lfloor N_t \rfloor}}{\sqrt{N}}$$

W_t is defined on t=$[0, \infty)$ by letting $W_t^N = \frac{X_{\lfloor N_t \rfloor}}{\sqrt{N}}$
W_t denotes a stochastic process termed, the Wiener process.

By definition, a process $Z(t)$ is a Wiener process if it satisfies the following properties:

i. Independence: $W_t - W_s$ is independent of $\{X_\tau\}_{\tau \leq s}$, for any $0 \leq s \leq t$

ii. Stationarity: The statistical distribution of $W_{t+s} - W_s$ is independent of s

iii. Gaussianity: W_t is a Gaussian process with mean and covariance

$EW_t = 0, \; EW_t W_s = \min(t, s)$

iv. Continuity: With probability 1, W_t viewed as a function of t is continuous.

3.3.3 Ito process

An Ito process or a stochastic integral is a stochastic process on (Ω, F, P) adapted to F_t, which can be written in the form:
$X_t = X_0 + \int_0^t Us \; ds + \int_0^t Vs \; dBs$, where $U, V \in l_2$ which can be written as:

$$dX_t = Ut \; dt + Vt \; dBt$$

Thus, B_t^2 is an Ito process:

$$B_t^2 = \int_0^t ds + 2 \int_0^t Bs \; dBs$$

or

$$(B_t^2)^2 = dt + 2Bst \; dBt$$

The term dt arises because Brownian motion B, is not differentiable and instead has quadratic variation (Fall, 2013).
If a stochastic X_t is defined to be an Ito process with respect to B_t if there exists $Ut \in l_2$ and $V_{i,t} \in l_2$, $1 \leq i \leq d$ such that $X_t = Ut \; dt + \varepsilon_{i,t} \; dB_{i,t}$

3.3.4 Ito formula for one-dimensional Brownian Motion

Let Xt be an Ito process:

$$dX_t = U_t\ dt + V_t\ dB_t$$

Suppose $g(x) \in c^2(2)$ is twice continuously differentiable function (in particular all second partial derivatives are continuous functions)

Suppose $(X_t) \in l_2$, then,
$Y_t = g(X_t)$ ia an Ito process and $dY_t = \frac{\partial g}{\partial x}(X_t)\ dX_t + \frac{1}{2}\frac{\partial^2 g}{2\partial x^2}(X_t)(dX_t)^2$

Using the notational convention for $dX_t = U_t\ dt + V_t\ dB_t$ and $(dX_t)^2$, the Ito formula can be rewritten as: $dY_t = \left(dY_t = \frac{\partial g}{\partial x}(X_t)U_t + \frac{1}{2}\frac{\partial^2 g}{2\partial x^2}(X_t)(V_t)^2\right) dt$

Hence, the space of Ito process is closed under twice-continuously differentiable transformations (Fall, 2013).

3.3.5 Multidimensional Ito formula

Suppose $dX_t = U_t\ dt + V_t\ dB_t$, where $U = (U_1, ..., U_d)$ and matrix $V = (V_{11}, .., V_{dd})$ have l_2 components and B is the vector of d independent Brownian motions. Let $g(x)$ be twice continuously differentiable function from R^d into R, then: $Y_t = g(X_t)$ is an Ito process and:

$$dY_t = \sum_{i=1}^{d} \frac{\partial g}{\partial x_i}(X_t)dX_{i,t} + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 g}{\partial x_i \partial x_j}(X_t)dx_{i,t}\dot{d}x_{j,t}$$

Where $dx_{i,t}\dot{d}x_{j,t}$ is computed using the following rules:

i. $dtdt = dtdB_i = dB_idt = 0$

ii. $dB_idB_j = 0$ for all$i \neq j$ and $(dB_i)^2 = dt$

3.3.6 Black-Scholes equation

Suppose that the arbitrage free market contains the following:

i. An underlying security whose price is governed by a Geometric Brownian motion; $dS(t) = S(t)\mu dt + S(t)\sigma dZt$, process over a time interval [0,T], where μ and σ are constants.

ii. A risk free asset with dynamics $dB(t) = rB(t)dt$, where the interest rate r is constant.

iii. A simple contingent claim of $\chi = \Phi(S(t))$ which can be traded on the market with the price $\prod(t)$

Then, the only pricing equation of the form $\prod(t) = F(t, S(t))$ which is consistent with the absence of arbitrage is when F satisfies the following partial differential equation (PDE):

$$\frac{\partial F}{\partial t}(t,s) + rs\frac{\partial F}{\partial s}(t,s) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2}(t,s) - rF(t,s) = 0$$

subject to the boundary condition: $F(T, s) = \Phi(s)$ in the strip $[0, T]$. This PDE is called the Back-Scholes-Merton model (Hosseini, 2007)

3.3.7 Feynman-Kac theorem

Feynman-Kac theorem

Suppose that x_t follows the stochastic process: $dx_t = \mu(x_t, t)dt + \sigma(x_t, t)dW_t^Q$, where W_t^Q is a Brownian motion under the measure Q.

Let $V(x_t, t)$ be a differentiable function of x_t and t and suppose that $V(x_t, t)$ follows the PDE given by:

$$\frac{\partial V}{\partial t} + \mu(xt,t)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma(xt,t)^2\frac{\partial^2 V}{\partial x^2} - r(x_t,t)V(x_t,t) = 0$$

and with boundary condition $V(X_T, T)$.

The theorem asserts that $V(x_t, t)$ has the solution

$$V(x_t, t) = E^Q\left[e^{\int_t^T r(x_u,u)du} V(X_T, T)|F_t\right]$$

The generator of the process in $dx_t = \mu(x_t, t)dt + \sigma(x_t, t)dW_t^Q$ is defined as the operator:

$$A = \mu(x_t, t)\frac{\partial}{\partial x} + \frac{1}{2}\sigma(x_t, t)^2\frac{\partial}{\partial x^2}$$

Hence the PDE in $V(x_t, t)$ can be written as:

$$\frac{\partial V}{\partial t} + AV(x_t, t) - r(x_t, t)V(x_t, t) = 0$$

Multidimensional version of the Feynman-Kac theorem

Suppose Xt follows the stochastic equation $dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t^Q$, where X_t and $\mu(X_t, t)$ are vectors of dimension n. W_t^Q is a vector of dimension m of Q-Brownian motion, and $\sigma(X_t, t)$ is a matrix of size $n \times m$.

i.e. $$d\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} \mu_1(t) \\ \vdots \\ \mu_n(t) \end{pmatrix} dt + \begin{pmatrix} \sigma_1 1(x_t, t) & \cdots & \sigma_1 m(x_t, t) \\ \vdots & \ddots & \vdots \\ \sigma_n 1(x_t, t) & \cdots & \sigma_n m(x_t, t) \end{pmatrix} \begin{pmatrix} dW_1^Q(t) \\ \vdots \\ dW_m^Q(t) \end{pmatrix}$$

The generator of the process is:

$$A = \sum_{i=1}^{n} \mu_i \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{i=1}^{n}(\sigma\sigma)_{i,j}^T \frac{\partial^2}{\partial x_i \partial x_j}$$

Where $\mu_i = \mu_i(X_t, t), \sigma = \sigma(X_t, t)$ and $(\sigma\sigma)_{i,j}^T$ is element (i, j) of matrix $(n \times n)$.

The theorem states that the PDE in $V(X_t, t)$ is given by:

$$\frac{\partial V}{\partial t} + AV(X_t, t) - r(X_t, t)V(X_t, t) = 0$$

and with boundary condition $V(X_T, T)$ has solution:

$$V(X_t, t) = E^Q \left[e^{\int_t^T r(X_u, u)du} V(X_T, T)|F_t \right]$$

3.3.8 Ornstein-Uhlenbeck process

A stochastic process $\{X_t : t \geq 0\}$ is an Ornstein-Uhlenbeck process if it is stationary, Gaussian, Markovian and continuous in probability.
Generally, $\{X_t : t \geq 0\}$ satisfies the linear stochastic differential equation:

$$dX_t = \rho(X_t - \mu)dt + \sigma dW_t$$

where $\{W_t : t \geq 0\}$ is a Brownian motion and μ, ρ and σ are constants and in an unconditional (strictly stationary) case, we have $E(X_t) = \mu$ and $\mathrm{cov}(X_s, X_t) = \frac{\sigma^2}{2\rho} e^{-\rho|s-t|}$ (Hosseini, 2007)

3.3.9 Risk neutral valuation

Consider a market given by the following equations:

$$dB = rB(t)dt \tag{3.1}$$

$$dS(t) = s(t)\alpha(t, S(t)) + S(t)\sigma(t, S(t))dW(t) \tag{3.2}$$

Equation (3.2) denotes the P-dynamics of the S-process, with a contingent claim $\chi = \Phi(S(t))$.

The arbitrage free market is given by: $\prod(t, \Phi) = F(t, S(t))$, where F is a solution of the Blacks-Scholes pricing equation.

By applying the Feynman-Kac theorem, the solution n is given by: $F(t,s) = e^{-r(T-t)} E_{t,s}[\Phi(X(T))]$, X is a process defined by:

$$dX(u) = rX(u)du + X(u)\sigma(u, X(u))dW(u) \tag{3.3}$$

Where $X(t) = s$ and W is a Brownian motion.

The price process of S in equation (3.2) is similar to the price of X in equation (3.3), with a difference in their drifts. Hence, there exists another probability measure, Q, under which the S-process is described by the SDE:

$$dS(t) = rS(t)dt + S(t)\sigma(t, S(t))dW(t) \tag{3.4}$$

where W is a Brownian motion with respect to Q.
Equation (3.4) is called the representation of S, the Q-dynamics of S.

Let E^Q be the expectation under the martingale probability measure, Q.
Considering the Q-dynamics, the following equation can be obtained:

$$F(t,s) e^{-r(T-t)} E^Q_{t,s}[\Phi(X(T))] \tag{3.5}$$

The Q-measure is called the risk-adjusted measure and equation (3.5) is called the risk neutral valuation formula. (Hosseini, 2007)

3.3.10 The pricing equation

The pricing equation is premised on the following assumptions.

- a k-dimensional stochastic process $dX(t) = \mu(t, X(t))dt + \partial(t, X(t))dW(t)$, dW is an n-dimensional Wiener process

- risk free asset: $dB(t) = rB(t)dt$
- there is a liquid market for all contingent claims written on X
- the claims $y_i = \Phi^i(X(T)),\ i = 1, 2, 3, ...$ are chosen and at their price processes, are $\prod_i(t) = F_i(t, X(t))$ with $F^i(T, x) = \Phi^i(x),\ i = 1, 2, 3, ...$

By Ito's formula, $dF^i = F^i\alpha_i dt + F^i\alpha_i d\bar{W}$
Where, $\sigma_i = \dfrac{F_t^i + (F_x^i)^*\mu\frac{1}{2}tr[\delta^* F_{xx}^i\delta]}{F^i}$ and $\sigma_i = \dfrac{(F_x^i)\delta}{F^i}$

Let $\sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{bmatrix}$ be an invertible matrix at each point (t, x)

- another T-claim $y = \Phi(X(T))$ with $\prod = F(x, X(t))$
- $F(T, x) = \Phi(x)$.

By Ito's formula, $dF^i = F^i\alpha_F dt + F^i\alpha_F d\bar{W}$
Where, $\alpha_F = \dfrac{F_t + F_x^*\mu\frac{1}{2}tr[\delta^* F_{xx}\delta]}{F}$ and $\sigma_F = \dfrac{(F_x^*)\delta}{F^i}$

If the market is arbitrage free, then the process:
$\lambda = \sigma^{-1} \begin{bmatrix} \alpha_1 - r \\ \alpha_2 - r \\ \vdots \\ \alpha_n - r \end{bmatrix}$ Satisfies the identity $\alpha_F - r = \sigma_F\lambda$.
λ is called "risk premium per unit of volatility.

COROLLARY

1. Under the above assumptions, F is a solution of:

$$F_t + F_x^*\mu - \delta\lambda + \frac{1}{2}tr[\delta^* F_{xx}\delta] = rF$$

$$F(T, x) = \Phi(x)$$

2. In the above model, there exists a martingale measure Q and an n-dimensional Wiener process W with respect to Q such that:

$$F(t,x)e^{-r(T-t)}E^{Q}_{t,x}[\Phi(X(T))]$$

Moreover, $dX = (\mu - \delta\lambda)dt + \sigma dW$.

3.3.11 Stochastic Integrals

$$L^2[a,b] = \left\{ f : \int_a^b E[f(s)]ds < \infty \text{ and } f(s) \in F_t^W \right\}$$

If $f \in L^2[a,b]$, then,

$$E\left[\int_a^b f(u)dW_u\right] = 0,$$

$$var\left[\int_a^b f(u)dW_u\right] = \int_a^b E[f^2(u)]du$$

$$\begin{aligned} cov\left(\int_a^b f(u)dW_u, \int_a^b g(u)dW_u\right) &= E\left[\left(\int_a^b f(u)dW_u\right)\cdot\left(\int_a^b g(u)dW_u\right)\right] \\ &= E\left[\left(\int_a^b f(u)dW_u\right)\cdot\left(\int_a^b g(u)dW_u\right)\right] \\ &= \int_a^b E\left[f(u)g(u)\right]du \end{aligned}$$

where f is a stochastic process and W_u is a Wiener process.

3.4 One-Factor Modeling of Crude Oil Prices

The one factor model uses spot price as the primary factor in determining futures prices of crude oil. The one-factor model's assumptions are based on the dynamics of the spot price. The spot price is modeled as a Geometric Brownian Motion (GBM) process or as a Mean-Reversion process.

A widely used one–factor model was developed by Brennan and Schwartz (1985).

In this model, the spot price is assumed to follow the Geometric Brownian motion

$$dS(t) = \mu S(t)dt + \sigma_S S dz \tag{3.6}$$

Where:

$S(t)$ is the spot price

μ is the drift of the spot price

σ_S is the volatility of the spot price

dz is a Wiener process associated with $S(t)$

Brennan and Schwartz (1985) show that the solution to their one factor model is given by the following Feynman–Kac solution:

$$F(S,t,T) = Se^{(r-c)}\tau \tag{3.7}$$

Hayat (2013) observed that the simplicity of this model is tractable. Hence, it does not capture the influence of producers and consumers in the commodities market. When spot price is high, producers tend to increase their production rate and consumers tend to use their surplus stocks, thus reducing the spot price.

Also when spot price is low, consumers increase their stocks and producers decrease their production rate causing the price of the commodity to increase. Thus, several one- factor models assume that spot prices follow a mean reverting process. Schwartz (1997) models the spot price as:

$$dS(t) = kS(t)(\mu - ln(S))dt + \sigma_S S(t)dz \tag{3.8}$$

Where:

$\mu =$ the long run mean of spot price

$k =$ the rate of mean reversion

The futures price $F(S,T)$ satisfies the following equation

$$\ln F(S,T) = e^{-kt} \ln S(T) + \left(1 - e^{-kT}\right) \mu + \frac{\sigma_S^2}{4k} \left(1 - e^{-2kT}\right) \tag{3.9}$$

3.5 Two-Factor Modeling of Crude Oil Prices

The one-factor models have been shown to be simplistic for modeling commodities term structure since they are based on only spot price. Also the one- factor mean-reverting model was not satisfactory because all futures returns were correlated which do not give a realistic solution in modeling the futures and spot prices of commodities. To improve on the performance of the one factor –models, several two factor-models were developed. The most widely used two-factor model is the one proposed by Schwartz (1997). All other two-factor models are based on this model. The Schwartz (1997) model is based on the spot prices and conference yields.

Assumptions of model

- S is the spot price of oil described by the stochastic process:

$$dS(t) = \mu - \delta S(t)dt + \sigma_1 S(t)dz_1 \tag{3.10}$$

- δ is the instantaneous convenience yield described by the Urnstein-Uhlenbeck process:

$$d\delta(t) = k(\bar{\delta} - \delta)dt + \sigma_2 dz_2 \tag{3.11}$$

Where:

μ is the long-term- total return on oil

k is the mean reverting coefficient

δ is the long–term convenience yield σ_1 and σ_2 are the volatilities of the spot price of oil and the convenience yield

dz_1 and dz_2 are the Wiener processes associated with $S(t)$ and $\delta(t)$ respectively.

$$z_1(t) \text{ and } z_2(t) \sim N(0, t^{\frac{1}{2}}) \tag{3.12}$$

Where $t\frac{1}{2}$ represents the standard deviation

$$dz_1 \cdot dz_2 = \rho dt \tag{3.13}$$

for the two correlated stochastic processes.

The Brownian motion with respect to the martingale probability measure is given as:

$$dz_1 = dz_1^* - \frac{\mu - r}{\sigma_1} dt \tag{3.14}$$

$$dz_2 = dz_2^* - \lambda dt \tag{3.15}$$

$\frac{\mu - r}{\sigma_1}$ is the market price per unit of oil and λ is the market price of convenience yield risk.

Substituting equations (14) and (15) into equations (10) and (11), the following equations are obtained

$$dS(t) = (r - \delta)S(t)dt + \sigma_1 S(t) dz_1^* \tag{3.16}$$

$$d\delta(t) = \left[k(\bar{\delta} - \delta) - \sigma_2 \lambda\right] dt + \sigma_2 dz_2^* \tag{3.17}$$

The price of the oil is assumed to be a twice continuously differentiable function of S and δ (Gibson and Schwartz, 1990)

By defining $Y(t) = F(t, s, \delta)$ and applying Ito's lemma for the correlated process to equations (11) and (12), the following result is obtained.

$$dY = \Big\{F_t + (r-\delta)SF_S + \left[k(\bar{\delta}-\delta) - \sigma_2\lambda\right]F_\lambda + \frac{1}{2}\sigma_1^2 S^2 F_{SS}$$

$$+\frac{1}{2}\sigma_2^2 F_{\delta\delta} + \sigma_1\sigma_2\rho S F_{S\delta}\Big\} + F_S S\sigma_1 dz_1^* + F\delta\sigma_2 dz_2^* \tag{3.18}$$

This shows an instantaneous price change. The current price of a futures contract, $F(t,s,\delta)$, on one barrel of crude oil satisfies the following PDE which is a two-dimensional pricing equation.

$$dY = F_t + (r-\delta)SF_S + \left[k(\bar{\delta}-\delta) - \sigma_2\lambda\right]F_\lambda + \frac{1}{2}\sigma_1^2 S^2 F_{SS} + \frac{1}{2}\sigma_2^2 F_{\delta\delta} + \sigma_1\sigma_2\rho S F_{S\delta} = rF \tag{3.19}$$

$$F(t,s,\delta) = S(T)$$

$$F(t,x) = e^{-r(T-t)}E_{t,x}^{Q}[\Phi(X(T))]$$

Hence,

$$F(t,s,\delta) = e^{-r(T-t)}E_{t,s,\delta}^{Q}[\Phi(X(T))] \tag{3.20}$$

$S(T)$ is given as:

$$S(T) = S(t)\cdot\exp\Big\{-\frac{1}{2}\sigma_1^2 + r - \bar{\delta} + \frac{\lambda\sigma_2}{k}(T-t) + \sigma_1\int_t^T dz_1^* - \frac{\sigma_2}{k}\int_t^T dz_2^* +$$

$$\frac{\sigma_2}{k}e^{-kT}\int_t^T e^{ks}dz_2^* + \frac{1}{k}\left(1-e^{-k(T-t)}\right)\left(\frac{-\lambda\sigma_2}{k} + \bar{\delta} - \delta(t)\right)\Big\} \tag{3.21}$$

Inserting equation (16) into equation (15) and simplifying further, the following equation is obtained.

$$F(t,s,\delta) = E^Q\Big[S\cdot\exp\Big\{-\frac{1}{2}\sigma_1^2 + r - \bar{\delta} + \frac{\lambda\sigma_2}{k}(T-t) + \sigma_1\int_t^T dz_1^* - \frac{\sigma_2}{k}\int_t^T dz_2^* +$$

$$\frac{\sigma_2}{k}e^{-kT}\int_t^T e^{ks}dz_2^* + \frac{1}{k}\left(1-e^{-k(T-t)}\right)\left(\frac{-\lambda\sigma_2}{k} + \bar{\delta} - \delta(t)\right)\Big\} \tag{3.22}$$

$$F(t,s,\delta) \equiv E^Q[S\cdot\exp\{D\}]$$

The current price of the futures contract is given as:

$$E^Q[S \cdot \exp\{D\}] = S \cdot \exp\{\hat{\mu} + \frac{1}{2}\hat{\sigma}^2\}$$

$$\Rightarrow S{\cdot}\exp\{\hat{\mu}{+}\frac{1}{2}\hat{\sigma}^2 = S{\cdot}\exp\left\{\left(-\bar{\delta} + \frac{\lambda\sigma_2}{k} + \frac{1}{2}\frac{\sigma_2^2}{k^2} - \frac{\rho\sigma_1\sigma_2}{k}\right)(T-t) + \frac{1}{k}\left(-\frac{\lambda\sigma_2}{k} + \delta - \bar{\delta}(t) + \right.\right.$$

$$\left.\frac{\rho\sigma_1\sigma_2}{k} - \frac{\sigma_2^2}{k^2}\right)\left(1 - e^{-k(T-t)}\right) + \left(\frac{\sigma_2}{k}\right)^2\left(\frac{1}{4k}\right)\left(1 - e^{-2k(T-t)}\right) \quad (3.23)$$

The futures price of a futures contract, $F^f(t, s, \delta)$, on one barrel of crude oilis obtained by multiplying the obtained "current price" by $e^{r(T-t)}$, where $T - t$ is the to maturity.

Hence,

$$F^f(T-t, s, \delta) = e^{r(T-t)} \cdot F(t, s, \delta)$$

$$= S \cdot \exp\left\{\left(r - \bar{\delta} + \frac{\lambda\sigma_2}{k} + \frac{1}{2}\frac{\sigma_2^2}{k^2} - \frac{\rho\sigma_1\sigma_2}{k}\right)(T-t) + \frac{1}{k}\left(-\frac{\lambda\sigma_2}{k} + \delta - \bar{\delta}(t) + \right.\right.$$

$$\left.\frac{\rho\sigma_1\sigma_2}{k} - \frac{\sigma_2^2}{k^2}\right)\left(1 - e^{-k(T-t)}\right) + \left(\frac{\sigma_2}{k}\right)^2\left(\frac{1}{4k}\right)\left(1 - e^{-2k(T-t)}\right) \quad (3.24)$$

This represents the price of a futures contract on one barrel of crude oil.

3.6 The Parsimonious Two-Factor Model of Crude Oil Future

The parsimonious two-factor model was developed and implemented by Cortazar and Schwartz (2003). This model is a modification of the two-factor model presented by Schwartz (1997) in other to give a more parsimonious representation of the two-factor model.

The two-factor parsimonious model has the following advantages over other two-factor models.

- It is simple because of fewer parameters
- It does not include the estimation of the risk-free interest rate from bond data
- The model is more intuitive for practitioners as returns are defined in terms of the long-term price appreciation instead of long-term convenience yield.

Assumptions:

- y is the bemeaned δ subtracted from the long term convenience yield.
  i.e.
  $$y = \delta - \bar{\delta} \tag{3.25}$$
- v is the long-term price appreciation on oil obtained by subtracting the long-term convenience yield from the long-term total return.
  i.e.
  $$v = \mu - \delta \tag{3.26}$$

Recall:

From the two-factor Schwartz (1997) model,

$$dS = (\mu - \delta)Sdt + \sigma_1 Sdz_1 \tag{3.27}$$
$$d\delta = k(\bar{\delta} - \delta)dt + \sigma_2 dz_2 \tag{3.28}$$

Substituting equations (25) and (26) into equations (27) and (28), we obtain,

$$dS = (v - y)Sdt + \sigma_1 Sdz_1 \tag{3.29}$$
$$d\delta = -kydt + \sigma_2 dz_2 \tag{3.30}$$

In formulating this model, both S and y are treated as non-traded state variables. We therefore transform the original processes into the risk-adjusted processes by assigning one risk premium to each process. This is known as the risk-neutral valuation. If λ_1 and λ_2 are the risk premiums, the, we have:

$$dS = (v - y - \sigma_1 \lambda_1) S dt + \sigma_1 S dz_1 \tag{3.31}$$

$$d\delta = (-ky - \sigma_2 \lambda_2) dt + \sigma_2 dz_2 \tag{3.32}$$

and

$$dz_1^* \cdot dz_2^* = \rho dt \tag{3.33}$$

This parsimonious two-factor model is the basis of the three-factor model as indicated by Cortazar and Schwartz (2003).

3.7 The Three-Factor Model of Crude Oil Futures Prices

Based on the parsimonious two-factor modeling of crude oil futures prices, Cortazar and Schwartz (2003) again developed a three-factor model for crude oil futures prices. In this model, they considered the long-term spot price return as a third factor, allowing it to be stochastic and to mean-revert to a long-term average. The other two factors they considered are the stochastic spot price and the short-term stochastic convenience yield.

Model Assumptions:

- The spot price, S, depends on the bemeaned convenience yield (y) and the long-term total return (v).

  i.e.

$$dS = (v - y) S dt + \sigma_1 S dz_1 \tag{3.34}$$

- The bemoaned convenience yield (y) is described by an Ornstein-Uhlenbeck process, meaning it is mean-reverting.
  i.e.

$$dy = -kydt + \sigma_2 dz_2 \tag{3.35}$$

- The long-term spot price (v) return (price appreciation) is also an Ornstein-Uhlenbeck process.
  i.e.

$$dv = a(\bar{v} - v)dt + \sigma_3 dz_3 \tag{3.36}$$

Parameters:

k = the mean-reverting coefficient a = the mean reverting coefficient for the third factor. v = the expectation of the long-term spot price return. σ_1, σ_2 and σ_3 are the volatilities of the spot price of oil, bemeaned, convenience yield and the long-term spot price return respectively.

From the assumptions above, the three correlated stochastic processes obtained are as follows:

$$dz_1 dz_2 = \rho_{12} dt \tag{3.37}$$

$$dz_1 dz_3 = \rho_{13} dt \tag{3.38}$$

$$dz_2 dz_3 = \rho_{23} dt \tag{3.39}$$

The cumulative bemeaned convenience yield rate and the expected cumulative long-term spot price from the date 0 to the date t, are defined by the following equations:

$$X(t) = \int_0^t y(x)dx \tag{3.40}$$

$$L(t) = \int_0^t v(x)dx \tag{3.41}$$

Also to show the relation between the Brownian motion z_1, z_2, z_3 with respect to the true probability measure and martingale probability measure are defined as follows:

$$dz_i = dz_i^* - \lambda_i dt, \tag{3.42}$$

$i = 1, 2, 3, ...$ and λ_i is price of risk.

$$dS = (v - y - \sigma_1\lambda_1)Sdt + \sigma_1 S dz_1^* \tag{3.43}$$

$$dy = (-ky - \sigma_2\lambda_2)dt + \sigma_2 dz_2^* \tag{3.44}$$

$$dv = a(\bar{v} - v) - \sigma_3\lambda_3 dt + \sigma_3 dz_3^* \tag{3.45}$$

We define:

$$Y(t) = F(t, s, \delta, v)$$

Applying Ito's lemma to the 3-correlated processes (equations 43, 44 and 45), we obtain the following result:

$$dY = \{f_t + (v - y - \sigma_1\lambda_1)SF_S + (-ky - \sigma_2\lambda_2)F_y + [a(\bar{v} - v) - \sigma_3\lambda_3]F_V +$$

$$S\sigma_1\sigma_2\rho_{12}F_{Sy} + \sigma_2\sigma_3\rho_{23}F_{yV} + S\sigma_1\sigma_3\rho_{13}F_{SV} + \frac{1}{2}S^2\sigma_1^2F_{SS} + \frac{1}{2}\sigma_2^2F_{yy} + \frac{1}{2}\sigma_3^2F_{VV}\Big\} dt$$

$$+ S\sigma_1F_S dz_1^* + \sigma_2F_y dz_2^* + \sigma_3F_V dz_3^* \tag{3.46}$$

This shows instantaneous price change.

The current price a futures contract, $F(t, s, y, v)$ on one barrel of crude oil, satisfies the following PDE.

$$f_t + (v - y - \sigma_1\lambda_1)SF_S + (-ky - \sigma_2\lambda_2)F_y + [a(\bar{v} - v) - \sigma_3\lambda_3]F_V + S\sigma_1\sigma_2\rho_{12}F_{Sy} +$$

$$\frac{1}{2}S^2\sigma_1^2F_{SS} + \frac{1}{2}\sigma_2^2F_{yy} + \frac{1}{2}\sigma_3^2F_{VV}\sigma_2\sigma_3\rho_{23}F_{yV} + S\sigma_1\sigma_3\rho_{13}F_{SV} = rF \tag{3.47}$$

Subject to the boundary condition:

$$F(t, s, y, v) = S(T)$$

Recall:

$$F(t, s, y, v) = e^{-r(T-t)} \cdot E^{Q}_{t,s,y,v}[\Phi(S(T))] \tag{3.48}$$

Where:

$$\Phi(S(T)) = S(T)$$

3.7.1 Evaluating S(T)

Recall:

$$ds(u) = (v - y)Sdu + \sigma_1 S dz_1 \tag{3.49}$$

and

$$S(T) = S$$

also,

$$d(\log S) = odu + \frac{1}{S}dS + \frac{1}{2}(-\frac{1}{S^2})(dS)^2 \tag{3.50}$$

Substituting dS in the above equation, the following is obtained:

$$d(\log S) = (v - y - \frac{1}{2}\sigma_1^2)du + \sigma_1 dz_1 \tag{3.51}$$

Where:

$$dz_1 = dz_1^* - \lambda_1 du \tag{3.52}$$

$$\Rightarrow S(T) = S(t)\exp\left\{\int_t^T v(s)ds - \int_t^T y(s)ds - \frac{1}{2}\sigma_1^2(T-t) + \sigma_1\int_t^T dz_1^* - \sigma_1\lambda_1(T-t)\right\} \tag{3.53}$$

But

$$\int_t^T v(s)ds \equiv L(T) - L(t) \quad \text{and} \quad \int_t^T dv(s) = V(T) - V(t) \tag{3.54}$$

Also

$$dz_3 = dz_3^* - \lambda_3 dt \tag{3.55}$$

Hence

$$\int_t^T dv = a\bar{v}(T-t) - a\int_t^T vds + \sigma_3 \int_t^T dz_3^* - \sigma_3\lambda_3(T-t) \tag{3.56}$$

$$\therefore\ V(T) - V(t) = a\bar{v}(T-t) - a\left(L(T) - L(t)\right) + \sigma_3 \int_t^T dz_3^* - \sigma_3\lambda_3(T-t) \tag{3.57}$$

Hence

$$V(T) = \bar{v}\left(1 - e^{-a(T-t)}\right) + e^{-aT}\sigma_3 \int_t^T dz_3^*\ e^{as} + \frac{\sigma_3\lambda_3}{a}\left(1 - e^{-a(T-t)}\right) \tag{3.58}$$

$$\Rightarrow \qquad L(T) - L(t) = -\frac{v}{a}\left(1 - e^{-a(T-t)}\right) + \frac{v}{a}\left(1 - e^{-a(T-t)}\right) -$$

$$\frac{1}{a}e^{-aT}\sigma_3 \int_t^T dz_3^*\ e^{as}\frac{\sigma_3\lambda_3}{a^2}\left(1 - e^{-a(T-t)}\right) + \bar{v}(T-t) + \frac{\sigma_3}{a}\int_t^T dz_3^* - \frac{\sigma_3\lambda_3}{a}(T-t) \tag{3.59}$$

We also recall that:

$$\int_t^T y(s)ds \equiv X(T) - X(t) \tag{3.60}$$

$$\int_t^T dy(s) = y(T) - y(t) \tag{3.61}$$

$$dz_2 = dz_2^* - \lambda_2 dt \tag{3.62}$$

$$\Rightarrow \qquad \int_t^T dy = -k\int_t^T yds + \sigma_2 \int_t^T dz_2^* - \sigma_2\lambda_2(T-t) \tag{3.63}$$

Giving rise to the following equation

$$y(T) - y(t) = -k\left(X(T) - X(t)\right) + \sigma_2 \int_t^T dz_2^* - \sigma_2\lambda_2(T-t) \tag{3.64}$$

$$\Rightarrow \qquad y(T) = e^{-k(T-t)}yt + e^{-kt}\sigma_2 \int_t^T dz_2^*\ e^{ks} - \frac{\sigma_2\lambda_2}{k}(1 - e^{-k(T-t)}) \tag{3.65}$$

$$\therefore X(T) - X(t) = \frac{y}{k}(1 - e^{-k(T-t)}) - \frac{1}{k}e^{-kt}\sigma_2 \int_t^T dz_2^*\ e^{ks}\frac{\sigma_2\lambda_2}{k^2}(1 - e^{-k(T-t)}) +$$

$$\frac{\sigma_2}{k}\frac{t}{T}dz_2^* - \frac{\sigma_2\lambda_2}{k}(T-t) \tag{3.66}$$

Substituting equations (59) and (66) into equation (53) and rearranging terms, we obtain the following equation:

$$S(T) = S(t)\exp\left\{-\frac{v}{a}(1-e^{-a(T-t)}) + \frac{v}{a}(1-e^{-a(T-t)}) - \frac{1}{a}e^{-aT}\sigma_3\int_t^T dz_3^*\ e^{as} + \right.$$

$$\frac{\sigma_3\lambda_3}{a^2}(1-e^{-a(T-t)}) + \bar{v}(T-t) + \frac{\sigma_3}{a}\int_t^T dz_3^* - \frac{\sigma_3\lambda_3}{a}(T-t) - \frac{y}{k}(1-e^{-k(T-t)}) +$$

$$\frac{\sigma_2}{k}e^{-kt}\int_t^T dz_2^* e^{ks} - \frac{\sigma_2\lambda_2}{k^2}(1-e^{-k(T-t)}) - \frac{\sigma_2}{k}\int_t^T dz_2^* + \frac{\sigma_2\lambda_2}{k}(T-t) - \frac{1}{2}\sigma_1^2(T-t) +$$

$$\left.\sigma_1\int_t^T dz_1^* - \sigma_1\lambda_1(T-t)\right\} \tag{3.67}$$

Substituting equation (67) into equation (48) and simplifying, the following results is obtained

$$F(t,s,y,v) = e^{-a(T-t)}\cdot E^Q\left[S\exp\left\{-\frac{v}{a}(1-e^{-a(T-t)}) + \frac{v}{a}(1-e^{-a(T-t)}) - \right.\right.$$

$$\frac{1}{a}e^{-aT}\sigma_3\int_t^T dz_3^*\ e^{as} + \frac{\sigma_3\lambda_3}{a^2}(1-e^{-a(T-t)}) + \bar{v}(T-t) + \frac{\sigma_3}{a}\int_t^T dz_3^* - \frac{\sigma_3\lambda_3}{a}(T-t) -$$

$$\frac{y}{k}(1-e^{-k(T-t)}) + \frac{\sigma_2}{k}e^{-kt}\int_t^T dz_2^*\ e^{ks} - \frac{\sigma_2\lambda_2}{k^2}(1-e^{-k(T-t)}) - \frac{\sigma_2}{k}\int_t^T dz_2^* +$$

$$\left.\frac{\sigma_2\lambda_2}{k}(T-t) - \frac{1}{2}\sigma_1^2(T-t) + \sigma_1\int_t^T dz_1^* - \sigma_1\lambda_1(T-t)\right\} \tag{3.68}$$

$$\Rightarrow \quad F(t,s,y,v) = e^{-a(T-t)}\cdot E^Q\left[S\exp\left\{-\frac{v}{a}(1-e^{-a(T-t)}) + \frac{v}{a}(1-e^{-a(T-t)}) - \right.\right.$$

$$\frac{1}{a}e^{-aT}\sigma_3\int_t^T dz_3^*\ e^{as} + \frac{\sigma_3\lambda_3}{a^2}(1-e^{-a(T-t)}) + \bar{v}(T-t) + \frac{\sigma_3}{a}\int_t^T dz_3^* - \frac{\sigma_3\lambda_3}{a}(T-t) -$$

$$\frac{y}{k}(1-e^{-k(T-t)}) + \frac{\sigma_2}{k}e^{-kt}\int_t^T dz_2^*\ e^{ks} - \frac{\sigma_2\lambda_2}{k^2}(1-e^{-k(T-t)}) - \frac{\sigma_2}{k}\int_t^T dz_2^* +$$

$$\left.\frac{\sigma_2\lambda_2}{k}(T-t) - \frac{1}{2}\sigma_1^2(T-t) + \sigma_1\int_t^T dz_1^* - \sigma_1\lambda_1(T-t)\right\} - r(T-t) \tag{3.69}$$

$$F(t,s,y,v) = E^Q\left[Sexp\{M\}\right] \tag{3.70}$$

To calculate the above expectation, we need to calculate the expectation and variance of M as follows:

$$\hat{\sigma_2} = var^Q[M]$$

$$\hat{\mu} = -\frac{v}{a}(1-e^{-a(T-t)}) + \frac{v}{a}(1-e^{-a(T-t)})\frac{\sigma_3\lambda_3}{a^2}(1-e^{-a(T-t)}) + \bar{v}(T-t)\frac{\sigma_3\lambda_3}{a}(T-t) -$$
$$\frac{y}{k}(1-e^{-k(T-t)}) - \frac{\sigma_2\lambda_2}{k^2}(1-e^{-k(T-t)}) - \frac{\sigma_2\lambda_2}{k}(T-t) - \frac{1}{2}\sigma_1^2(T-t) - \sigma_1\lambda_1(T-t) - r(T-t) \tag{3.71}$$

Also

$$\hat{\sigma_2} = E^Q[M2] - (EQ[M])^2$$

$$\hat{\sigma^2} = E^Q\left[\left(-\frac{1}{a}e^{-aT}\sigma_3\int_t^T dz_3^* e^{as}\right) + \left(\frac{\sigma_3}{a}\int_t^T dz_3^*\right) + \right.$$
$$\left.\left(\frac{\sigma_2}{k}e^{-kt}\int_t^T dz_2^*\, e^{ks}\right) + \left(-\frac{\sigma_2}{k}\int_t^T dz_2^*\right) + \left(-\frac{\sigma_2}{k}\int_t^T dz_2^*\right)\right]^2 \tag{3.72}$$

Taking several expectations, we have the following result.

$$\hat{\sigma^2} = \frac{1}{a^2}\sigma_3^2\frac{1}{2a}\left(1-e^{-k(T-t)}\right) + \frac{\sigma_3^2}{a^2}(T-t) + \left(\frac{\sigma_2}{k}\right)^2\cdot\frac{1}{2k}\left(1-e^{-k(T-t)}\right) + \frac{\sigma_2^2}{k^2}(T-t) +$$

$$\sigma_1^2(T-t) - \frac{2\sigma_3^2}{a^2}\left(1-e^{-k(T-t)}\right) - \frac{2\sigma_2\sigma_3\rho_{23}}{ak}\cdot\frac{1}{a+k}\left(1-e^{-(a+k)(T-t)}\right) + \frac{2\sigma_2\sigma_3\rho_{23}}{a^2k}\cdot\left(1-e^{-k(T-t)}\right)$$
$$\frac{2\sigma_1\sigma_3\rho_{13}}{a^2}\left(1-e^{-k(T-t)}\right) + \frac{2\sigma_2\sigma_3\rho_{23}}{ak^2}\left(1-e^{-k(T-t)}\right) - \frac{2\sigma_2\sigma_3\rho_{23}}{ak}(T-t) + \frac{2\sigma_1\sigma_3\rho_{13}}{a}(T-t) -$$
$$\frac{2\sigma_2^2}{k^2}\left(1-e^{-k(T-t)}\right) + \frac{2\sigma_1\sigma_2\rho_{12}}{k^2}\left(1-e^{-k(T-t)}\right) - \frac{2\sigma_1\sigma_2\rho_{12}}{k}(T-t) \tag{3.73}$$

Hence,

$$E^Q[S\exp M] = S\exp\left\{\hat{\mu} + \frac{1}{2}\hat{\sigma^2}\right\} \tag{3.74}$$

$$EQ[Sexp\{M\}] = S\exp\left\{(T-t)\left(\bar{v} - \frac{\sigma_3\lambda_3}{a} + \frac{\sigma_2\lambda_2}{k} - \sigma_1\lambda_1 + \frac{\sigma_3^2}{2a^2} + \frac{\sigma_2^2}{2a^2} + \frac{\sigma_1\sigma_3\rho_{13}}{a}\right.\right.$$
$$\left. -\frac{\sigma_2\sigma_3\rho_{23}}{ak} - \frac{\sigma_1\sigma_2\rho_{12}}{k} - r\right) + (1-e^{-a(T-t)})\left(-\frac{v}{a} + \frac{v}{a} + \frac{\sigma_3\lambda_3}{a^2} - \frac{\sigma_3^2}{a^3} + \frac{\sigma_2\sigma_3\rho_{23}}{a^2k} - \frac{2\sigma_1\sigma_3\rho_{13}}{a^2}\right)$$
$$+(1-e-2aT-t)\left(\frac{\sigma_3^2}{4a^3}\right) + (1-e^{-k(T-t)}) - \left(\frac{y}{k} - \frac{\sigma_2\lambda_2}{k^2} + \frac{\sigma_2\lambda_3\rho_{23}}{ak^2} - \frac{\sigma_2^2}{k^3} + \frac{\sigma_1\lambda_2\rho_{12}}{k^2}\right) +$$

$$(1-e^{-2k(T-t)})\left(\frac{\sigma_2^2}{4k^3}\right)-\left(1-e^{-(k+a)(T-t)}\right)\left(\frac{\sigma_2\lambda_3\rho_{23}}{ak(a+k)}\right)\Bigg\} \tag{3.75}$$

This represents the current price of a futures contract on one barrel of crude oil.

The future price of a futures contract, FfT-t,s,y,v, on one barrel of crude oil is given below.

$$F^f(T-t,s,y,v)=e^{r(T-t)}\cdot F(t,s,y,v) \tag{3.76}$$

$$F^f(T-t,s,y,v)=S\exp\Bigg\{(T-t)\left(\bar{v}-\frac{\sigma_3\lambda_3}{a}+\frac{\sigma_2\lambda_2}{k}-\sigma_1\lambda_1+\frac{\sigma_3^2}{2a^2}+\frac{\sigma_2^2}{2a^2}+\frac{\sigma_1\sigma_3\rho_{13}}{a}\right.$$

$$\left.-\frac{\sigma_2\sigma_3\rho_{23}}{ak}-\frac{\sigma_1\sigma_2\rho_{12}}{k}\right)+(1-e^{-a(T-t)})\left(-\frac{v}{a}+\frac{v}{a}+\frac{\sigma_3\lambda_3}{a^2}-\frac{\sigma_3^2}{a^3}+\frac{\sigma_2\sigma_3\rho_{23}}{a^2k}-\frac{2\sigma_1\sigma_3\rho_{13}}{a^2}\right)$$

$$+(1-e-2aT-t)\left(\frac{\sigma_3^2}{4a^3}\right)+\left(1-e^{-k(T-t)}\right)-\left(\frac{y}{k}-\frac{\sigma_2\lambda_2}{k^2}+\frac{\sigma_2\lambda_3\rho_{23}}{ak^2}-\frac{\sigma_2^2}{k^3}+\frac{\sigma_1\lambda_2\rho_{12}}{k^2}\right)+$$

$$(1-e^{-2k(T-t)})\left(\frac{\sigma_2^2}{4k^3}\right)-\left(1-e^{-(k+a)(T-t)}\right)\left(\frac{\sigma_2\lambda_3\rho_{23}}{ak(a+k)}\right)\Bigg\} \tag{3.77}$$

This represents the price of crude oil futures contract on one barrel of crude oil when time to maturity is $T-t$.

3.7.2 The Volatility Term Structure of Futures Returns

Assuming $\tau = T-t$, to determine the volatility of futures returns, we apply Ito's lemma to the above equation of futures contract.

The following is obtained:

$$dF^f(\tau,s,y,v)=F_S^f dS+F_S^f dy+F_S^f dv-F_\tau^f dt+\frac{1}{2}F_{SS}^f(dS)^2 \tag{3.78}$$

But:

$$\begin{cases} F_S^f \cdot dS &= \dfrac{F^f}{S} \cdot ((v-y)Sdt + \sigma_1 S dz_1) \\ F_y^f \cdot dy &= F^f \cdot \left(-\dfrac{1-e^{-k\tau}}{k}\right)(-kydt + \sigma_2 dz_2) \\ F_v^f \cdot dv &= F^f \cdot \left(-\dfrac{1-e^{-k\tau}}{k}\right)(a(\bar{v}-v)dt + \sigma_3 dz_3) \\ & \text{and} \\ F_{SS}^f &= 0. \end{cases}$$

Hence,

$$dF^f(\tau, s, y, v) = \frac{F^f}{S} \cdot ((v-y)Sdt + \sigma_1 S dz_1) + F^f \cdot \left(-\frac{1-e^{-k\tau}}{k}\right)(-kydt + \sigma_2 dz_2) +$$

$$F^f \cdot \left(-\frac{1-e^{-k\tau}}{k}\right)(a(\bar{v}-v)dt + \sigma_3 dz_3) - F^f \cdot u(t)dt \tag{3.79}$$

From the above we obtain the following:

$$\sigma_F f^2(\tau) = E\left(\frac{dF^f}{F^f}\right)^2 - E^2\left(\frac{dF^f}{F^f}\right) \tag{3.80}$$

Whereas $E^2\left(\frac{dF^f}{F^f}\right) = 0$

$$\Rightarrow \qquad \left(\frac{dF^f}{F^f}\right)^2 = (\sigma_1 dz_1)^2 + \left(-\frac{1-e^{-a\tau}}{k}\right)^2 (\sigma_2 dz_2)^2 + \left(\frac{1-e^{-a\tau}}{a}\right)(\sigma_3 dz_3)^2 -$$

$$2\left(\frac{1-e^{-a\tau}}{k}\right)(\sigma_1 dz_1)(\sigma_2 dz_2) - 2\left(\frac{1-e^{-a\tau}}{k}\right)\left(\frac{1-e^{-a\tau}}{a}\right)(\sigma_2 dz_2)(\sigma_3 dz_3) +$$

$$2\left(\frac{1-e^{-a\tau}}{a}\right)(\sigma_1 dz_1)(\sigma_3 dz_3) \tag{3.81}$$

$$\left(\frac{dF^f}{F^f}\right)^2 = E\left[(\sigma_1^2 dt)^2 + \left(-\frac{1-e^{-a\tau}}{k}\right)^2 (\sigma_2^2 dt)^2 + \left(\frac{1-e^{-a\tau}}{a}\right)(\sigma_3^2 dt)^2 -\right.$$

$$2\left(\frac{1-e^{-a\tau}}{k}\right)\sigma_1\sigma_2\rho_{12}dt - 2\left(\frac{1-e^{-a\tau}}{k}\right)\left(\frac{1-e^{-a\tau}}{a}\right)\sigma_2\sigma_3\rho_{23}dt +$$

$$2\left(\frac{1-e^{-a\tau}}{a}\right)\sigma_1\sigma_3\rho_{13}dt \tag{3.82}$$

From (80) and the last equality, the following result is obtained

$$\sigma_F f^2 = \sigma_1^2 + \left(-\frac{1-e^{-a\tau}}{k^2}\right)\sigma_2^2 + \left(\frac{1-e^{-a\tau}}{a^2}\right)\sigma_3^2 - 2\left(\frac{1-e^{-a\tau}}{k}\right)\sigma_1\sigma_2\rho_{12} -$$

$$2\left(\frac{1-e^{-a\tau}}{k}\right)\left(\frac{1-e^{-a\tau}}{a}\right)\sigma_2\sigma_3\rho_{23} + 2\left(\frac{1-e^{-a\tau}}{a}\right)\sigma_1\sigma_3\rho_{13} \tag{3.83}$$

When $\tau \longrightarrow \infty$,

$$\lim_{\tau\longrightarrow\infty} \sigma_F^2 f = \sigma_1^2 + \frac{\sigma_2^2}{k^2} + \frac{\sigma_3^2}{a^2} - 2\frac{\sigma_1\sigma_2\rho_{12}}{k} - 22\frac{\sigma_2\sigma_3\rho_{23}}{ak} + 2\frac{\sigma_1\sigma_3\rho_{13}}{a}$$

$$\lim_{\tau\longrightarrow\infty} \sigma_F^2 f = \left(\sigma_1 - \frac{\sigma_2}{k} + \frac{\sigma_3}{a}\right)^2 \tag{3.84}$$

This shows that the volatility of futures returns converges to a positive constant as $\tau \longrightarrow \infty$

3.8 Hedging Crude Oil Price Fluctuations

Hedging is a risk management strategy used to limit or offset the likelihood of a loss from fluctuations in the prices of commodities, currency or securities. Many nations, either exporters or importers of crude oil have explored various possibilities of hedging crude oil price volatilities as part of measures to insulate their economies and citizens against the dangers associated with these price fluctuations. Crude oil futures and options contracts are the two main strategies that have been employed to hedge crude oil price risks.

Crude Oil futures are standardized, exchange-traded contracts in which the contract buyer agrees to take delivery, from the seller, a specific quantity of crude oil (e.g. 1000 barrels) at a predetermined price on a future delivery date. Consumers and producers of crude oil can manage crude oil price risk by purchasing and selling crude oil futures. Crude Oil producers can employ a short hedge to lock in a selling price for the crude oil they produce while businesses that require crude oil can utilize a long hedge to secure a purchase price for the commodity they need. Crude Oil futures are also traded by speculators who assume the price risk that hedgers try to avoid in return for a chance to profit from favorable crude oil price movements. Speculators buy crude oil futures when they believe that crude oil prices will go up. Conversely, they will sell crude oil futures when they think that crude oil prices will fall.

Crude Oil options are option contracts in which the underlying asset is a crude oil futures contract. The holder of a crude oil option possesses the right (but not the obligation) to assume a long position (in the case of a call option) or a short position (in the case of a put option) in the underlying crude oil futures at the strike price. This right will cease to exist when the option expires after market closes on expiration date.

3.9 Crude Oil Options versus Crude Oil Futures

Options have the following advantages over futures:

1. Compared to taking a position on the underlying crude oil futures outright, the buyer of a crude oil option gains additional leverage since the premium payable is typically lower than the margin requirement needed to open a position in the underlying crude oil futures.

2. As crude oil options only grant the right but not the obligation to assume the underlying crude oil futures position, potential losses are limited to only the premium paid to purchase the option.

3. Using options alone for hedging, or in combination with futures, a wide range of strategies can be implemented to cater for a specific risk profile, investment time horizon, cost consideration and outlook on underlying volatility.

4. Options have a limited lifespan and are subjected to the effects of time decay. The value of a crude oil option, specifically the time value, gets eroded away as time passes. However, since trading is a zero sum game, time decay can be turned into an ally if one chooses to be a seller of options instead of buying them.

However, crude oil options are also wasting assets that have the potential to expire worthless.

3.9.1 Types of Crude Oil Options Contracts

Options are divided into two classes - calls and puts. The buyer of a call option pays a premium to the seller and, in return, has the right (but not obligated) to buy a specific amount and type of oil at a fixed price, before or at a given date. The buyer of a put option pays a premium to the seller and, in return, has the right (but not obligated) to sell a specific amount and type of oil at a fixed price, before or at a given date. The fixed pre-determined price at which the holder of the option can buy or sell the underlying asset is usually known as the strike price or exercise price. The date agreed in the contracts is usually known as the expiration date, exercise date or the maturity of the option.

3.9.2 Option Contract Specifications

The following terms are specified in an option contract.

Option Type: Call options confer the buyer the right to buy the underlying stock while put options give him the rights to sell them.

Strike Price: The strike price is the price at which the underlying asset is to be bought or sold when the option is exercised. It's relation to the market value of the underlying asset affects the moneyness of the option and is a major determinant of the option's premium.

Premium: In exchange for the rights conferred by the option, the option buyer has to pay the option seller a premium for carrying on the risk that comes with the obligation. The option premium depends on the strike price, volatility of the underlying, as well as the time remaining to expiration.

Expiration Date: Option contracts are wasting assets and all options expire after a period of time. Once the stock option expires, the right to exercise no longer exists and the stock option becomes worthless. The expiration month is specified for each option contract. The specific date on which expiration occurs depends on the type of option. For instance, stock options listed in the United States expire on the third Friday of the expiration month.

Option Style: An option contract can be either American style or European style. The manner in which options can be exercised also depends on the style of the option. American style options can be exercised any time before expiration while European style options can only be exercised on expiration date itself. All of the stock options currently traded in the marketplaces are American-style options. Some option writers also prefer to sell European options as this allows them time to plan their exposure more accurately.

Underlying Asset: The underlying asset is the security which the option seller has the obligation to deliver to or purchase from the option holder in the event the option is exercised. In the case of stock options, the underlying asset refers to the shares of a specific company. Options are also available for other types of securities such as currencies, indices and commodities.

Contract Multiplier: The contract multiplier states the quantity of the underlying asset that needs to be delivered in the event the option is exercised.

For stock options, each contract covers 100 shares.

3.9.3 The Options Market

Participants in the options market buy and sell call and put options. Those who buy options are called holders. Sellers of options are called writers. Option holders are said to have long positions, and writers are said to have short positions.

The price paid or received for buying or selling an option is called premium. Premium on options can be split into two components. These are: intrinsic value and time value.

The intrinsic value is the difference between the underlying price and the strike price, to the extent that it is in favor of the option holder. For a call option, intrinsic value is given as the current stock price minus strike price. For a put option, the intrinsic value is equal to strike price minus current stock price.

That is:

$$\begin{aligned} \text{Intrinsic value} &= \text{current stock price} - \text{strike price (for call option)} \\ \text{Intrinsic value} &= \text{strike price} - \text{current stock price (for put option)} \end{aligned}$$

Time value is the amount the option trader is paying for a contract above its intrinsic value, with the belief that prior to expiration the contract value will increase because of a favorable change in the price of the underlying asset. The longer the amount of time until the expiration of the contract, the greater its time value.

Hence, Time value = option premium – intrinsic value

3.9.4 Crude Oil Options and futures Exchanges

Crude Oil option and futures contracts are mostly traded at New York Mercantile Exchange (NYMEX) and Tokyo Commodity Exchange (TOCOM).

NYMEX Light Sweet Crude Oil option prices are quoted in dollars and cents per barrel and their underlying futures are traded in lots of 1000 barrels (42000 gallons) of crude oil. NYMEX Brent Crude Oil options and futures are traded in contract sizes of 1000 barrels (42000 gallons) and their prices are quoted in dollars and cents per barrel.

TOCOM Crude Oil futures prices are quoted in yen per kiloliter and are traded in lot sizes of 50 kiloliters (13210 gallons).

3.10 Hedging Against Rising Crude Oil Prices

Governments, businesses as well as other crude oil consumers that buy crude oil in significant quantities can hedge against rising crude oil prices by taking up a position in the crude oil futures market. This involves buying crude oil call options to secure a purchase price for the supply of crude oil that they will require sometime in the future. To implement this long hedge, enough crude oil futures should be purchased to cover the quantity of crude oil required by the business operator or the government.

3.10.1 Example of Long Crude Oil Call Options

An investor observed that the near-month NYMEX Light Sweet Crude Oil futures contract is trading at the price of USD 40.30 per barrel. A NYMEX Crude Oil call option with the same expiration month and a nearby strike price of USD 40.00 is being priced at USD 2.6900/barrel. Since each underlying NYMEX Light Sweet Crude Oil futures contract represents 1000 barrels of crude oil, the premium he needs to pay to own the call option is USD 2,690.

Assuming that by option expiration day, the price of the underlying crude oil futures has risen by 15% and is now trading at USD 46.34 per barrel. At this price, his call option is now in the money.

3.10.2 Gain from Call Option Exercise

$$
\begin{aligned}
\text{Gain from Option} &= \text{(Market Price of Underlying Futures - Option Strike Price)} \\
&\quad \times \text{Contract Size} \\
&= \text{(USD 46.34/barrel - USD 40.00/barrel)} \\
&\quad \times \text{1000 barrels} \\
&= \text{USD 6,340} \\
\text{Investment} &= \text{Initial Premium Paid} \\
&= \text{USD 2,690} \\
\text{Net Profit} &= \text{Gain from Option Exercise – Investment} \\
&= \text{USD 6,340 - USD 2,690} \\
&= USD3,650 \\
\text{Return on Investment} &= 136\%
\end{aligned}
$$

By exercising his call option now, the investor gets to assume a long position in the underlying crude oil futures at the strike price of USD 40.00. This means that he gets to buy the underlying crude oil at only USD 40.00/barrel on delivery day.

To take profit, the investor enters an offsetting short futures position in one contract of the underlying crude oil futures at the market price of USD 46.35 per barrel, resulting in a gain of USD 6.3400/barrel. Since each NYMEX Light Sweet Crude Oil call option covers 1000 barrels of crude oil, gain from the long call position is USD 6,340. Deducting the initial premium of USD 2,690 he paid to buy the call option, his net profit from the long call strategy will come to USD 3,650.

3.11 Hedging Against Falling Crude Oil Prices

Crude Oil producers and exporters can hedge against falling crude oil price by taking up a position in the crude oil futures market. Producers and exporters can employ what is known as a short hedge to lock in a future selling price for an ongoing production of the commodity that is only ready for sale sometime in the future. To implement this hedge, crude oil producers sell (short) enough crude oil futures contracts in the futures market to cover the quantity of crude oil to be produced.

3.11.1 Example

An oil exporting company decides to go short one near-month NYMEX Brent Crude Oil Futures contract at the price of USD 44.20/barrel. Since each Brent Crude Oil futures contract represents 1000 barrels of crude oil, the value of the contract is USD 44,200. To enter the short futures position, the company has to put up an initial margin of USD 12,825. A week later, the price of crude oil falls and correspondingly, the price of NYMEX Brent Crude Oil futures drops to USD 39.78 per barrel. Each contract is now worth only USD 39,780. So by closing out its futures position now, the company can exit the short position in Brent Crude Oil Futures with a profit of USD 4,420. The return on this investment will be 34.46%.

That is:

SELL 1000 barrels of crude oil at USD 44.20/barrel	=	USD 44,200
BUY 1000 barrels of crude oil at USD 39.78/barrel	=	USD 39,780
Profit	=	4,420
Investment (Initial Margin)	=	USD 12,825
Return on Investment	=	34.46%

An alternative way of hedging against falling crude oil prices is by buying crude oil put options. For example the company observed that the near-month NYMEX Light Sweet Crude Oil futures contract is trading at the price of USD 40.30 per barrel. A NYMEX Crude Oil put option with the same expiration month and a nearby strike price of USD 40.00 is being priced at USD 2.6900/barrel. Since each underlying NYMEX Light Sweet Crude Oil futures contract represents 1,000 barrels of crude oil, the premium the company needs to pay to own the put option is USD 2,690.

Assuming that by option expiration day, the price of the underlying crude oil futures has fallen by 15% and is now trading at USD 34.25 per barrel. At this price, the company's put option is now in the money.

3.11.2 Gain from Put Option Exercise

$$
\begin{aligned}
\text{Gain from Option Exercise} &= (\text{Option Strike Price - Market Price of Underlying Futures}) \\
&\quad \times \text{Contract Size} \\
&= (\text{USD } 40.00/\text{barrel - USD } 34.25/\text{barrel}) \\
&\quad \times 1000 \text{ barrel} \\
&= \text{USD } 5{,}750 \\
\text{Investment} &= \text{Initial Premium Paid} \\
&= \text{USD } 2{,}690 \\
\text{Net Profit} &= \text{Gain from Option Exercise} - \text{Investment} \\
&= \text{USD } 5{,}750 \text{ - USD } 2{,}690 \\
&= \text{USD } 3{,}060 \\
\text{Return on Investment} &= 114\%
\end{aligned}
$$

By exercising its put option now, the company gets to assume a short position in the underlying crude oil futures at the strike price of USD 40.00. In other words, it also means that it gets to sell 1,000 barrels of crude oil at USD 40.00/barrel on delivery day. To take profit, the company enters an offsetting long futures position in one contract of the underlying crude oil futures at the market price of USD 34.26 per barrel, resulting in a gain of USD 5.7500/barrel. Since each NYMEX Light Sweet Crude Oil put option covers 1,000 barrels of crude oil, gain from the long put position is USD 5,750. Deducting the initial premium of USD 2,690 it paid to purchase the put option, the company's net profit from the long put strategy will come to USD 3,060.

3.12 Pricing and Hedging of Crude Oil Options

Before any crude oil investor ventures into options trading, it is imperative for him or her to have a good understanding of the factors that affect the value or premium of options. These factors include:

- The current (strike) price of oil
- The strike price
- Volatility of stock price
- The risk-free interest rate
- The dividends expected during the life of the option

3.12.1 Current Price and Stock Price

If a call option is exercised at some future time, the payoff will be the amount by which the stock price will exceed the strike price. Hull (2009) explains that call options become more valuable as stock price increases and less valuable as the strike price increases. For put options, the payoff on the exercise is the amount by which the strike price exceeds the strike price. Put options therefore become less valuable as the stock price increases and more valuable as strike price increases.

3.12.2 Time to Expiration

All options have a limited useful lifespan and every option contract is defined by an expiration month. The option expiration date is the date on which an options contract becomes invalid and the right to exercise it no longer exists. The option becomes worthless after expiration. American call and put options can be exercised any time before expiration while European call and put options can only be exercised on expiration date itself.

3.12.3 Volatility

The price of options on an underlying stock constantly change fluctuates with time. The degree by which the price of a stock fluctuates is termed as volatility. Volatility affects the trading of both call and put options as the price of the underlying stock is not stagnant but keeps changing. Hull (2009) explains that the owner of a call option benefits from price increases but has limited downside risk in the event of price decreases. Similarly the owner of a put option benefits from price decreases but has limited downside risk in the event of price increases (Hull, 2009).

3.12.4 Risk-free interest rate.

The risk-free interest rate affects the price of an option in a clear-cut way. As interest rate in the economy increases, the expected return required by an investor from a stock tends to increases. Also, the present value of any future cash flow received by the holder of the option decreases. The combined impact of these two factors according to Hull (2009) is to increase the value of call options and decrease the value of put options.

3.12.5 Amount of Future Dividends

Dividends as explained by Hull (2009) have the effect of reducing the stock price on the ex-dividend date. This is bad news for the value of a call option and good news for the value of put options. Hence, the value of a call option is negatively related to the size of an anticipated future dividend, whiles the value of a put option is positively related to the size of an anticipated future dividend.

3.12.6 Options Pricing Models

Several mathematical models have been used in the past few decades in options pricing theory. These include: The Black-Scholes-Merton model, the binomial options pricing model, Monte-Carlo options model, Heston model, e.t.c

Among the various options pricing models, the Back-Scholes-Merton and the Binomial models are the widely used. This study uses the Back-Scholes-Merton options pricing model to price options and value crude oil options futures. The Back-Scholes-Merton model unlike the other models can be used to calculate a very large number of options prices in a very short time.

3.12.7 The Back-Scholes-Merton Options Pricing Model

This model was introduced in 1973 in a paper entitled, "The Pricing of Options and Corporate Liabilities" published in the Journal of Political Economy. It was developed by three economists namely, Fischer Black, Myron Scholes and Robert Merton. The Black-Scholes-Merton model is regarded as the world's most well-known options pricing model.

Assume $\prod$ is a portfolio which consists of an option and a position in the underlying asset which hedges the future prices movements of the underlying stock.

Let $\prod = V - \delta \cdot St$

Since the stock price has lognormal dynamics it satisfies the SDE:

$$dS_t = rS_t dt + \sigma S_t dS_t$$

Where:

From Ito's lemma, the change in portfolio is given by:

$$\begin{aligned}
\prod &= V - \delta \cdot St & (3.85)\\
&= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS_t + \frac{\partial^2 V}{\partial S^2}(dS_t)^2 - \Delta \cdot dS_t & (3.86)\\
&= \left(\frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \left(\frac{\partial V}{\partial S} - \Delta\right) dS_t & (3.87)
\end{aligned}$$

Let $\Delta = \dfrac{\partial V}{\partial S}$

The change in value of the portfolio is no longer sensitive to changes in the underlying stock.

Thus:

$$d\prod = \left(\frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$$

This means that the return of the portfolio to leading order in time is certain. More specifically, there is no risk in the return depending on price changes of the underlying asset S_t by our choice of Δ. By the principle of no arbitrage the return of the portfolio must be the same as the risk-free interest rate r. Hence,

$$d\prod = r\prod dt$$

$$= r(V - \Delta \cdot S_t)dt$$

$$= r\left(V - S\frac{\partial V}{\partial S}\right)$$

From the above assumptions,

$$\left(\frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt = r\left(V - S\frac{\partial V}{\partial S}\right)$$

Simplifying further we obtain the following PDE:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

This PDE holds in the domain $t < T$ with the boundary condition $V(T, S) = f(S)$ given at maturity time T where the payoff $f(S)$ of the option is known. This equation is referred to as the Black-Scholes-Merton PDE.

3.12.8 Estimating the B-S-M PDE

Numerical and Monte-Carlo methods can be used to estimate the B-S-M PDE. This research uses the Implicit Crank-Nicolson numerical method to estimate the PDE. This method is unconditionally stable. The PDE is firstly transformed to the standard heat equation. The heat equation is then solved and transformed back to the solution of the PDE.

Let the transformation, $(s, x) \longrightarrow (x, \tau)$ be given by $s = Ke^x$ and $t = T - \frac{2\tau}{\sigma^2}$

If $V(x, \tau) = KV(s, t)$, then

$$\Rightarrow \qquad \frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (c-1)\frac{\partial v}{\partial x} - cv$$

Where $\qquad c = \frac{2\tau}{\sigma^2}$

If $v(x,\tau) = e^{-\frac{1}{2}(c-1)x-\frac{1}{4}(c+1)\tau}u(x,\tau)$, Then:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2}$$

The boundary condition of the B-S-M PDE which is the payoff at maturity,

$V(x,T) = Q(s)$, becomes the boundary condition $u(x,0) = \frac{1}{2}e^{-\frac{1}{2}(c-1)^2+(c+1)^2}Q(Ke^T)$ for the transformed system.

An exact solution to the basic heat equation can be written in terms of the initial conditions as:

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(u-x)^2}{4t}}u(y,0)dy$$

The approximate solution can be obtained from the implicit Crank-Nicolson method which is given as:

$$\frac{U_m^{n+1} - U_m^n}{\Delta\tau} = \frac{1}{2}\frac{U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}}{\Delta x^2} + \frac{1}{2}\frac{U_{m+1}^n - 2U_m^n + U_{m-1}^n}{\Delta x^2}$$

CHAPTER 4

ANALYSIS

4.1 Introduction

In this chapter, we have a simulation graph of the historical monthly prices of WTI crude oil for 2004 – 2014 periods. This graph shows that crude oil prices fluctuated within the period with the highest prices being recorded in 2008 and lowest prices recorded in 2014. We also have a graph of crude oil futures prices for the next decade as well as the graph of nominal prices and risk-adjusted prices. Numerical Simulation results of the three-factor modeling of crude oil futures prices and simulation results of the expected value of crude oil option using implicit Crank-Nicolson method.

4.2 Crude Oil Price Fluctuations Results and Analysis

Figure 4.1 represents the trend of crude oil prices in the last decade, that is, from 2004 to 2014. The study shows that crude oil prices rose in 2004 from $50per barrel to $60per barrel in august, 2005. By the middle of 2006, crude oil prices appreciated to $75per barrel and dropped to $60per barrel in early part of 2007. Prices however increased in October from $92 per barrel to $99.29 per barrel in December, 2007. The prices rose to historic height in June and July, 2008, trading at $141.71 per barrel and $147.02 per barrel respectively. This marginal increment was attributed to the global economic crises in 2008. Prices started falling from August, 2008 but rose again to $130 per barrel in September, 2008 and subsequently declined marginally to $60 per barrel at the end of 2008. It is observed that prices went slightly higher in 2009 but started falling again till the first half of 2010. We observed also that prices appreciated again at the end of December, 2010 selling at $ 100 per barrel. We also observed that oil prices again rose at the middle of June, 2011 but dropped at the end of the year. Appreciated again in the first quarter of 2012 but dropped steadily by the close of the year. In the first half of 2013, oil prices dropped slightly but increased at the end of the year. Between January, 2014 and June 2014, we observe that crude oil prices rose temporarily but dropped drastically at the end of 2014. The graph shows a continuous decline in crude oil prices from December, 2014.

Table 4.1 shows the monthly crude oil price history from 2004 to 2014. We observe that crude oil prices have undergone several swings in the last decade. This volatility is attributed to the demand and supply dynamics as well as political unrest in Iran, Libya, Ukraine and other oil producing nations. The volatility of the United States Dollar is a major contributory factor since it is the major currency traded in the international crude oil market.

YEARS	**MONTHLY PRICES IN U.S.A. DOLLARS**											
	Jan	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sept**	**Oct**	**Nov**	**Dec**
2004	34.31	34.69	36.74	36.75	40.28	38.03	40.78	44.90	45.94	53.28	48.47	43.15
2005	46.84	48.15	54.19	52.98	49.83	56.35	59.00	64.99	65.59	62.26	58.32	59.41
2006	65.49	61.63	62.69	69.44	70.84	70.95	74.41	73.04	63.80	58.89	59.08	61.96
2007	54.51	59.28	60.44	63.98	63.46	67.49	74.12	72.36	79.92	85.80	94.77	91.69
2008	92.97	95.39	105.45	112.58	125.40	133.88	133.37	116.67	104.11	76.61	57.31	41.12
2009	41.71	39.09	47.94	49.65	59.03	69.64	64.15	71.05	69.41	75.72	77.99	74.47
2010	78.33	76.39	81.20	84.29	73.74	75.34	76.32	76.60	75.24	81.89	84.25	89.15
2011	89.17	88.58	102.86	109.53	100.90	96.26	97.30	86.33	85.52	86.32	97.16	98.56
2012	100.27	102.20	106.16	103.32	94.66	82.30	87.90	94.13	94.51	89.49	86.53	87.86
2013	94.76	95.31	92.94	92.02	94.51	95.77	104.67	106.57	106.29	100.54	93.86	97.63
2014	94.62	100.82	100.80	102.07	102.18	105.79	103.59	96.54	93.21	84.40	75.79	59.29

Figure 4.1: Crude oil monthly price history ($/barrel) from 2004 to 2014

Against the backdrop of extreme oil price volatility, the global economic turmoil and other unforeseen economic crises, practitioners in the crude oil industry project crude oil futures prices. These price forecasts are based on perceptions of crude oil demand and supply in the short to long term. Figure 4.2 is a graph depicting crude oil price futures prices in the next decade. Since oil is the heart of the commodities market, these forecasts will enable both exporters and importers of the commodity to adopt measures of hedging against its price fluctuations so as to offset the dangers associated with these prices volatilities. We observed that despite the continuous downtrend in crude oil prices, analysts have projected a steady increase in the futures prices in the next decade, from a price of \$44.66 per barrel in April, 2015 to \$ 69.20 per barrel by December, 2023.

Figure 4.3 represents the annual nominal and risk-adjusted prices of crude oil prices from 2004 to 2014. A nominal price, sometimes called current dollar prices, is a measure of the dollar value of a product at the time it was produced. Real prices are adjusted when general price level changes over time, i.e., inflation or deflation. These adjustments give us a picture of prices for various years as if the value of the dollar was constant. The green line on the chat represents the risk-adjusted prices whiles the Blue line represents the nominal prices. We observed that June, 2008 recorded the highest inflation monthly adjusted average price of \$138.33/barrel. From there prices dropped significantly. In nominal terms, there was a fall in price from \$126.33/barrel in June, 2008 to \$31.04/barrel in February, 2009.in the run-up to 2008, the average price was nominally \$91.17/barrel and fell much lower to an average price of \$53.48/barrel. It then increased to \$61.46/barrel in 2010. By April, 2011, prices increased again but decreased drastically to \$76.90/barrel in September, 2011. The average price for 201 was \$87.04/barrel. In 2012, risk-adjusted prices were very much closed to the nominal average price of \$86.46/barrel. Prices rose slightly in 2013 to an average of \$91.17/barrel. The first 11 months of 2014 saw an average price of \$89.80/barrel.

We observed that the U.S. dollar had a tremendous influence on the real prices of oil. This is because crude is predominantly traded in dollars per barrel and any country except the U.S.A. will need to buy dollars to buy crude oil prices.

Table 4.2 shows the inflation rates of the USD for the period under consideration. Comparing the yearly inflation rates with the yearly crude oil prices, we observed that years with high inflation rates resulted in high oil prices. For example 2008 had a high inflation rate of 3.85% which corresponded to a high crude oil price of $138.33/barrel whiles 2009 with a low inflation rate of -0.34% resulting in a low crude oil price of $91.17/barrel.

Year	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014
Average inflation rate in %	2.68	3.39	3.24	2.85	3.85	-0.34	1.64	3.16	2.07	1.47	1.62

Table 4.2: average annual inflation rate of the USD

Figure 4.4 is a graph of four crude oil monthly futures traded at NYMEX for 2014. We observe that crude oil futures is often found in backwardation, which means higher prices for short term contracts than for long term contracts. Contract 1 prices rose from $34.22/barrel in January, 2004 to $53.09/barrel in November, but dropped to $48.48/barrel in December, 2004.

Figures 4.5 and 4.6 are the simulation results for the three-factor model of crude oil futures prices showing the factors: spot price, convenience yield and interest rate. The Blue lines represent the simulation paths whilst the Black lines represent the true mean of the various factors. We chose a small interval of 0.001 and a bigger time interval of 0.1 to ascertain the closeness of the simulation paths to the respective true mean of the various factors. We observed that the mean of 1000 simulation paths with a small time interval of 0.001 appears to be closer to the true mean than 100 simulations paths with a larger time interval of 0.1. This reveals that the simulation achieves better results when as many paths are used with smaller time interval. Table 4.3 shows the values of the parameters under the three-factor model.

Table 4.3: Values of parameters under three-factor model

K	y	a	$\hat{v}$	σ_1	σ_2	σ_3	ρ_1	ρ_2	ρ_3
0.3	1.0	0.18	0.76	0.25	0.15	0.1	0.24	0.3	0.08

Source: Tran (2010)

Figure 4.7 shows the options price surface of crude oil futures options traded at NYMEX. We plotted the Expected value of the option against the current value of the asset (Crude oil) and the time to maturity of the options contract. The payoff/expected value depends on both time and asset value. The surface has a greater slope due to constant volatility or constant change of crude oil prices. The expected value changes as time and asset prices also change. We observed that as the value of crude oil prices increase, the expected value of the option increases.

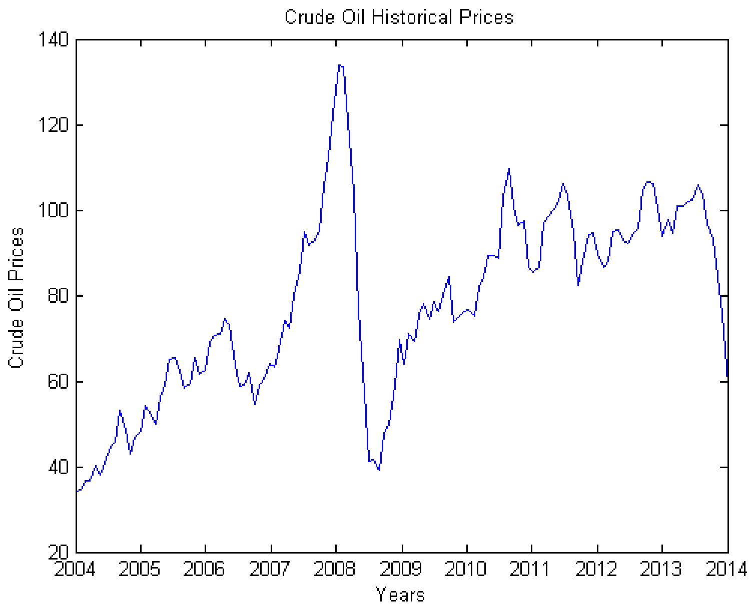

Figure 4.1: Crude oil prices history from 2004 to 2014

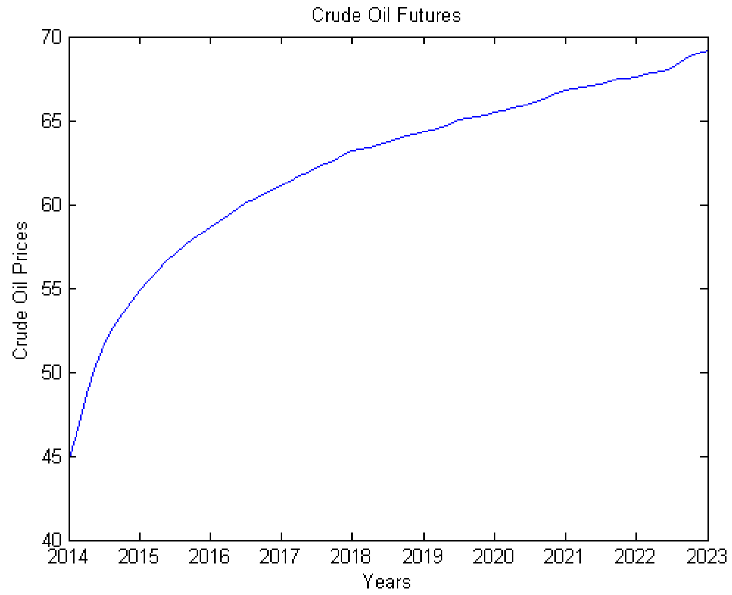

Figure 4.2: Crude oil futures prices from 2015 to 2023

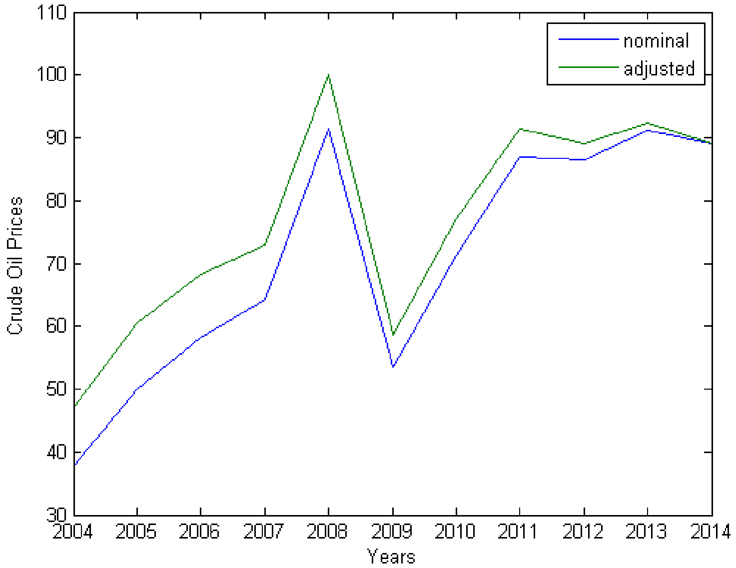

Figure 4.3: Nominal and Risk-adjusted prices

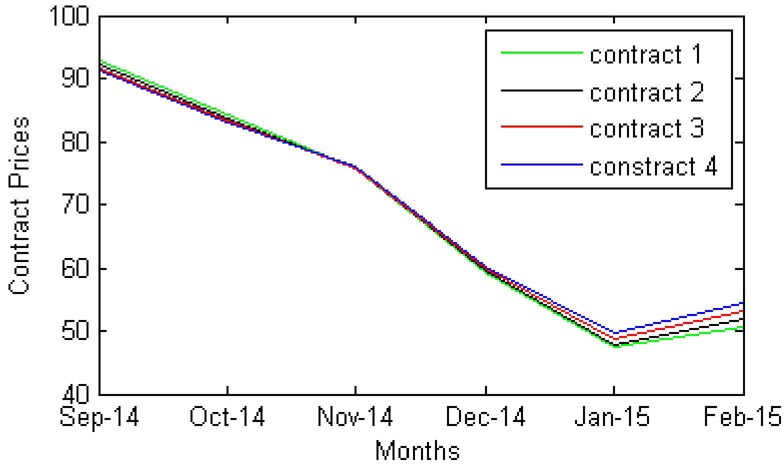

Figure 4.4: Crude oil contract prices for 2014 and 2015

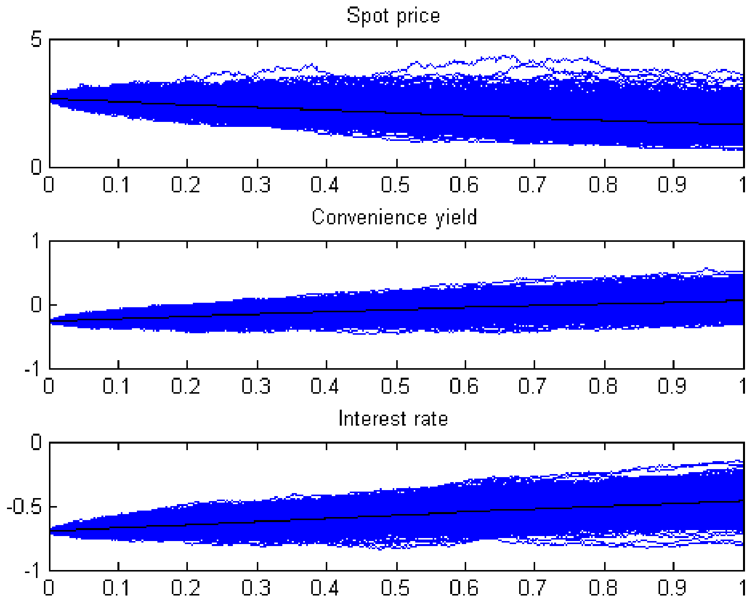

Figure 4.5: Simulation paths of 1000 with 0.001 time interval

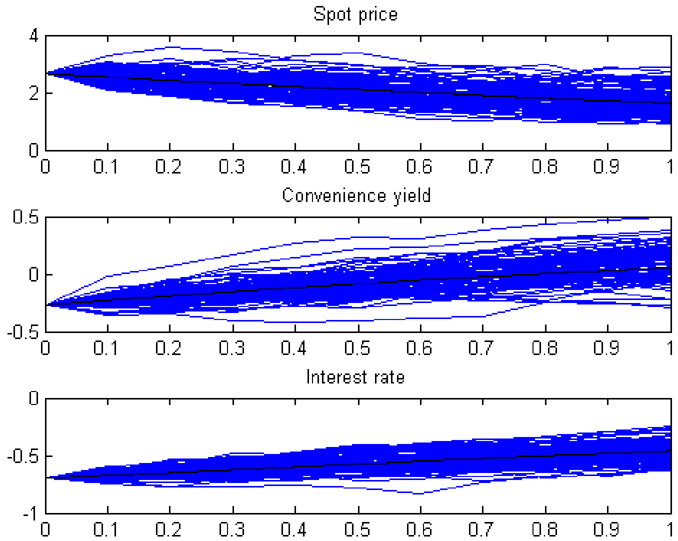

Figure 4.6: Simulation paths of 100 with 0.1 time interval

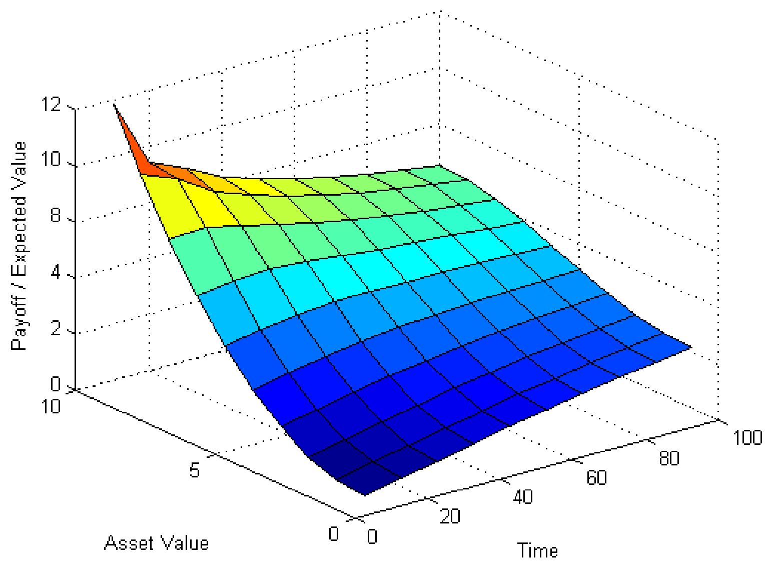

Figure 4.7: Surface of Crude Oil Options Expected Value

CHAPTER 5

CONCLUSION AND RECOMMENDATION

5.1 Conclusion

In this study we modeled crude oil futures prices as s stochastic process as proposed by Hosseini (2007). We concentrated much on the three-factor model. Analytical solutions were presented for this model based on the current price, the futures price and volatility of the commodity.

We studied the price dynamics of WTI crude oil traded at NYMEX from 2004 to 2014. Our study show that crude oil prices fluctuate thereby over the years with the highest price recorded in June, 2008 but dropped significantly to $41.12/barrel in December, 2008.

We also studied the price dynamics of crude oil futures prices and observed that for the period, April, 2015 to December, 2023. We noticed that despite the continuous fall in crude oil prices from November, 2014, futures prices increase continuously. This price increase is however not sharp but gradual.

We compared the nominal prices of crude oil to inflation risk-adjusted prices and observed that that the volatility of the USD has a greater influence on crude oil prices. Growth in inflation rate of the USD results in a corresponding increase in crude oil prices and vice versa. For instance a high inflation rate recorded in 2008 resulted in a marginal increase in crude oil prices in 2008. We further observed that nominal and risk adjusted prices grow at the same rate but converge after January, 2015.

Our simulations results on the three-factor model for crude oil futures prices revealed that a mean of 1000 simulation paths with a time interval of 0.001 appeared closer to the true mean that a smaller simulation paths of 100 with a larger time interval of 0.1. This indicates that the simulations achieve better results when as many paths with smaller time interval are used.

5.2 Recommendations

Crude oil price dynamics are very complex and unpredictable hence modeling these price behaviors is not an easy task. This behavior is influenced by many factors. Paramount among these factors is the volatility of the USD. We modeled crude oil prices using stochastic modeling and studied basic hedging strategies using options futures.

We recommend that future research on this topic should incorporate stochastic jumps in oil prices to model the crude oil price dynamics.

It is obvious that crude oil price volatility poses a major challenge to major economies. We therefore recommend that both exporters and importers of crude oil adopt measures to hedge crude oil prices so as to insulate their citizens against the risks associated with the price dynamics of oil in the global commodity market.

Finally, we recommend the use of the Datar–Mathews Method (DM method) for further research on options valuation. The DM Method provides an easy way to determine the real option value of a project simply by using the average of positive outcomes for the project. It was developed by Professor Vinay Datar of Seattle University and Scott H. Mathews, Technical fellow of the Boeing Company in the year 2000.

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Appendix A

```
load('data.mat','crude_oil_future_contract1','Wti_monthly_spot_prices',
'contracts1_4','Light_Sweet_Crude_Oil_Futures',
'monthly_crude_oil_prices_nominal_adjusted',
'WTI_crude_oil_historical_prices', 'yr', 'yr_nominal_adjusted')

figure
plot(yr,monthly_crude_oil_prices_nominal_adjusted)
xlabel('Years')
ylabel('Crude Oil Prices')
title('')
legend('nominal','adjusted')

figure
n=length(WTI_crude_oil_historical_prices);
xx=linspace(2004,2014,n);
plot(xx,WTI_crude_oil_historical_prices)
xlabel('Years')
ylabel('Crude Oil Prices')
title('Crude Oil Historical Prices')

figure
n=length(Light_Sweet_Crude_Oil_Futures);
xx=linspace(2014,2023,n);
plot(xx,Light_Sweet_Crude_Oil_Futures)
xlabel('Years')
ylabel('Crude Oil Prices')
```

```
title('Crude Oil Futures')

figure
xx=1:6;
plot(xx,contracts1_4(1,:),'g'),hold on
plot(xx,contracts1_4(2,:),'k')
plot(xx,contracts1_4(3,:),'r')
plot(xx,contracts1_4(4,:),'b')
xlabel('Months')
ylabel('Contract Prices')
months = ['Sep-14';'Oct-14';'Nov-14';'Dec-14';'Jan-15';'Feb-15'];
set(gca,'XTicklabel',months)
legend('contract 1','contract 2','contract 3','constract 4')

[u,x,t]=neumann_heat_cn(0,10,9,10);
surf(u)
xlabel('Time')
ylabel('Asset Value')
zlabel(' Payoff / Expected Value')
months = [20;40;60;80;100;120];
set(gca,'XTicklabel',months)
mm = [0; 2; 4; 8; 10; 12];
set(gca,'ZTicklabel',mm)


%Matlab Code - Generating the Three-factor
%Model
% The SDE of the 3-factor model (under the risk neutral framework) is:
% dS = (r-Delta).S.dt + sigma1.S.dZ1
% dDelta = kappa.(alpha-Delta).dt + sigma2.dZ2 (here alpha implies
```

```
% alpha_hat)
% dr = a.(m-r).dt + sigma3.dZ3 (here m implies m*)
% dZ1.dZ2 = rho12.dt, dZ1.dZ3 = rho13.dt, dZ2.dZ3 = rho23.dt

clear all
clc

randn('seed',1)
rand('seed',1)
T = 1;
sigma1 = 0.25; sigma2 = 0.15; sigma3 = 0.1;
m = 0.76; kappa = 0.3; a = 0.18; alpha = 1;
rho12 = 0.24; rho23 = 0.3; rho13 = 0.08;
S0 = 2*randn+0.7; Delta0 = randn; r0 = 2*randn+0.4;
Delta = [0.001 0.1];
g = 0;
Deta = 0.0001; t0 = [0:Deta:T];

for d = 1:500
XM1(1) = S0; XE2(1) = Delta0; XE3(1) = r0;
   for j = 1:length(t0)-1
      dW1 = sqrt(Deta)*randn;
      dW2 = sqrt(Deta)*randn;
      dW3 = sqrt(Deta)*randn;
      XM1(j+1) = XM1(j) + (XE3(j) - XE2(j))*XM1(j)*Deta ...
      + sigma1*XM1(j)*dW1 + 0.5*(sigma1^2)*XM1(j)*((dW1^2)-Deta);
      XE2(j+1)=XE2(j)+kappa*(alpha-XE2(j))*Deta+sigma2*rho12*dW1 ...
      + sigma2*sqrt(1-rho12^2)*dW2;
      XE3(j+1)=XE3(j)+a*(m-XE3(j))*Deta+sigma3*rho13*dW1 ...
```

```
      + sigma3*sqrt(1-rho13^2)*dW3;
     end
pat1(d,:) = XM1(:);
pat2(d,:) = XE2(:);
pat3(d,:) = XE3(:);
XM1=[]; XE2=[]; XE3=[];
end

truemean1 = mean(pat1(:,1:length(t0)));
truemean2 = mean(pat2(:,1:length(t0)));
truemean3 = mean(pat3(:,1:length(t0)));
pat1=[]; pat2=[]; pat3=[];

for k = 1:length(Delta)
  t = [];
  Xe1 = []; X1 = []; X2 = []; X3 = [];
  X1(1) = S0; Xe1(1) = S0;
  X2(1) = Delta0; X3(1) = r0;
  t = [0:Delta(k):T];

    if Delta(k) == 0.001
     d = 1000;
     elseif Delta(k) == 0.1
     d = 100;
    end
for u = 1:d
Xe1 = []; X1 = []; X2 = []; X3 = [];
Xe1(1) = S0; X1(1) = S0; X2(1) = Delta0; X3(1) = r0;
for j = 1:length(t)-1
```

```
dW1 = sqrt(Delta(k))*randn;
dW2 = sqrt(Delta(k))*randn;
dW3 = sqrt(Delta(k))*randn;
X1(j+1)=X1(j)+(X3(j)-X2(j))*X1(j)*Delta(k)+sigma1*X1(j)*dW1 ...
+ 0.5*(sigma1^2)*X1(j)*((dW1^2)-Delta(k));
X2(j+1)=X2(j)+kappa*(alpha-X2(j))*Delta(k)+sigma2*rho12*dW1 ...
+ sigma2*sqrt(1-rho12^2)*dW2;
X3(j+1)=X3(j)+a*(m-X3(j))*Delta(k)+sigma3*rho13*dW1 ...
+ sigma3*sqrt(1-rho13^2)*dW3;
end
if Delta(k) == 0.001
figure(1)
hold on
subplot(3,1,1),title('Spot price'),plot(t,X1)
hold on
subplot(3,1,2),title('Convenience yield'),plot(t,X2)
hold on
subplot(3,1,3),title('Interest rate'),plot(t,X3)
elseif Delta(k) == 0.1
figure(2)
hold on
subplot(3,1,1),title('Spot price'), plot(t,X1)
hold on
subplot(3,1,2),title('Convenience yield'), plot(t,X2)
hold on
subplot(3,1,3),title('Interest rate'), plot(t,X3)
end
g = g + 1
end
```

```
if Delta(k) == 0.001

figure(1)

hold on

subplot(3,1,1),title('Spot price'),plot(t0,truemean1,'k')

hold on

subplot(3,1,2),title('Convenience yield'),plot(t0,truemean2,'k')

hold on

subplot(3,1,3),title('Interest rate'),plot(t0,truemean3,'k')

elseif Delta(k) == 0.1

figure(2)

hold on

subplot(3,1,1),title('Spot price'),plot(t0,truemean1,'k')

hold on

subplot(3,1,2),title('Convenience yield'),plot(t0,truemean2,'k')

hold on

subplot(3,1,3),title('Interest rate'),plot(t0,truemean3,'k')

end

end

save threeMod -v7.3
```

Appendix B

CONTRACTS	MONTHS					
	September, 2014	October, 2014	November, 2014	December, 2014	January, 2015	February, 2014
Contract 1	93.03	84.34	75.81	59.29	47.33	50.72
Contract 2	92.18	83.72	75.83	59.57	47.90	51.82
Contract 3	91.67	83.31	75.84	59.88	48.72	53.13
Contract 4	91.39	82.98	75.89	60.25	49.62	54.45

Figure 5.1: NYMEX WTI Crude Oil Monthly Futures Contracts from September, 2014 to February, 2015 ($/barrel)

APR15 ---- 44.66	APR17 ----61.09	APR19 ---- 65.45	DEC22 ---- 69.20
MAY15 ---- 46.65	MAY17 ---61.39	MAY19 ---- 65.59	JUN23 ---- 69.20
JUN15 ---- 48.69	JUN17 ---61.71	JUN19 ---- 65.75	DEC23 ---- 69.20
JLY15 ---- 50.36	JLY17 ---61.88	JLY19 ---- 65.84	
AUG15 ---- 51.69	AUG17 ----62.11	AUG19 ---- 66.00	
SEP15 ---- 52.68	SEP17 ----62.37	SEP19 ---- 66.18	
OCT15 ---- 53.45	OCT17 ----62.63	OCT19 ---- 66.38	
NOV15---- 54.16	NOV17 ----62.90	NOV19 ---- 66.59	
DEC15 ---- 54.85	DEC17 ----63.17	DEC19 ---- 66.82	
JAN16 ---- 55.47	JAN18 ---63.28	JAN20 ---- 66.90	
FEB16 ---- 56.04	FEB18 ---63.40	FEB20 ---- 66.98	
MAR16 ---- 56.58	MAR18 ---63.54	MAR20 ---- 67.08	
APR16 ---- 57.08	APR18 ----63.70	APR20 ---- 67.19	
MAY16 ---- 57.52	MAY18 ---63.88	MAY20 ---- 67.32	
JUN16 ---- 57.94	JUN18 ---64.08	JUN20 ---- 67.46	
JLY16 ----58.29	JLY18 ---64.17	JLY20 ---- 67.53	
AUG16 ---- 58.64	AUG18 ----64.31	AUG20 ---- 67.62	
SEP16 ---- 58.99	SEP18 ----64.46	SEP20 ---- 67.72	
OCT16 ---- 59.35	OCT18 ----64.63	OCT20 ---- 67.84	
NOV16 ---- 59.72	NOV18 ----64.82	NOV20 ---- 67.98	
DEC16 ---- 60.09	DEC18 ----65.02	DEC20 ---- 68.12	
JAN17 ---- 60.32	JAN19 ----5.11	JUN21 ---- 68.49	
FEB17 ---- 60.56	FEB19 ---65.21	DEC21 ---- 68.85	
MAR17 ---- 60.81	MAR19 ---65.33	JUN22 ---- 69.03	

Figure 5.2: Light Sweet Crude Oil Futures Prices

Years	Nominal prices in USD	Risk-Adjusted prices in USD
2004	37.66	47.05
2005	50.04	60.45
2006	58.30	68.28
2007	64.20	72.99
2008	91.48	100.01
2009	53.48	58.76
2010	71.21	77.11
2011	87.04	91.39
2012	86.46	88.95
2013	91.17	92.41
2014	89.08	89.08

Figure 5.3: Nominal and Risk-Adjusted prices of crude oil ($/barrel)