Geodesics in (2 + 1)-dimensions

by

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Declaration

I hereby declare that this submission is my own work towards the MSc. And that, to the best of my knowledge, it contains no material previously published by another person or material which has been accepted for the award of any other degree of the university, except where due acknowledgement has been made in the text.

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ABSTRACT

The geodesics in (2+1) dimensional spacetime should be the same as the geodesics in (3+1) dimensional spacetime, since the two dimensional surface on which these geodesics lie should be embeddable 3-dimensional co-ordinate space. In the present thesis, we show precisely that .We demonstrate that the plane, the spherical surface, the ellipsoidal surface and the surface of a saddle on which these geodesics lie are embeddable in 3-dimensional co-ordinate space. We then go ahead and fine these geodesics. And clearly, the determination of the geodesics in (2+1) dimensional spacetime should be easier than in 3-dimensional spacetime since the number of equations involved in (2+1) dimensional spacetime is much smaller than in (3+1) dimensional spacetime. We talk about (2+1) and (3+1) dimensional spacetime because it is easier and more elegant to use the techniques of general relativity in the determination of these curves.

After showing that the surfaces indicated above are embeddable in 3-dimensional co-ordinate space, we go ahead and construct the 3-dimensional equivalent of the **Robertson-Walker metric.** The equation for the geodesics in general relativity is well known, and using our 3-dimensional metric, we compute all the geodesics on these surfaces which turn out to be the surfaces of zero, positive and negative curvatures

Not surprisingly, the geodesics in the plane and spherical surface were found to be straight line and great circles respectively. What can apparently be considered to be new results are the geodesics on ellipsoidal surface and the surface of a saddle which can really be described as 2-dimensional hyperbolic space. The geodesic on these last two surfaces were found to ellipses and hyperbolae. But it should be emphasized that in the relativistic language, curves are the geodesics in curved space and it is perhaps worth nothing that these curves are the trajectories of bodies attracted or repelled in force fields of the inverse square law

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Chapter 1

Introduction

Cosmology is the study of the large scale properties of the universe as a whole. Mathematical cosmology involves the formulation of theories that seem to explain existing astronomical data and make experimentally or observationally verifiable predictions. Cosmology makes statements about the whole universe. The approach to the study of cosmology is through the science of astronomy. Astronomy started as a study of the properties of planets and stars and gradually reached out to include the limits of the Milky Way system, which is our Galaxy

Modern astronomical techniques have taken the subject beyond the Galaxy to distant objects from which light may take billions of years to reach us. Cosmology is concerned mainly with this extragalactic world. Cosmology has three main aspects. They are **Cosmogony**: This is the study of the origin of the universe, **Cosmography**: That is cataloguing the objects in the universe and charting their positions and motions and **Theoretical Cosmology**: Here, we search for a framework within which to comprehend the information from cosmography. Theoretical cosmology employs the hypothesis known as the cosmological principle. The simplest model of the universe is obtained from the Cosmological Principle which states that the universe **is spatially homogeneous and isotropic**.

Homogeneity means there is no preferred point in space, so that we can locate the coordinate origin at any point in space without affecting physical laws. It affirms that all regions of sufficient size in space are equivalent and that physical laws are the same in all parts of the universe.

Isotropy means there is no preferred direction so orientation of the coordinate axes should have no effect on physical laws. This means that all directions in space are equivalent in regard to the formulation of fundamental physical laws. . (*Jayant* V Narliker)[1]

Translated into the language of Riemannian geometry, this cosmological principle asserts that three- dimensional coordinate space is a space of maximal symmetry or, a space of constant but possibly, time-dependent curvature.

In recent times, much attention has been given to the study of cosmology in (2+1)-dimensions than (3+1) -dimensions. This is apparently because computations in (2+1)-dimensions can be expected to be less complicated than computations in the higher (3+1)-dimensions which are being done to test gravitational theories, particularly quantum gravity. Our studies were motivated by this realization (namely, the realization that computations in (2+1)-dimensions could be easier than those in (3+1)-dimensions), which are to be independently of previous workers in the field. We also thought it would be interesting to determine some properties of the universe of the two-dimensional being (the Maxwell demon) and compare them with those of our universe.

This Thesis reports the determination of the geodesics (null and non- null in maximally symmetric 2-dimensional spaces of zero, positive and negative curvatures. As 3-dimensional coordinate space, the maximal symmetry leads to a Robertson-Walker-type metric, which, was to be expected, is simple but yields results that are valid for 3-dimensional coordinate space. These results will be discussed in the last chapter that is chapter five.

In chapter two we give some common definitions, mathematical tools for project (**tensor analysis**), and the definition of the Christofel symbols.

In chapter three, the general geodesic equation in tensorial form is derived. The curvature tensor and the Robertson-Walker metric in (3+1) - dimensions are also derived in this chapter.

Chapter four contains the actual work in which the derivation of the Robertson-Walker metric in (2+1) dimensions is carried out. The calculation for the curvature of this metric, the calculations of all the requisite Christoffel symbols, the formulation of the differential equations using the general geodesic equation and the appropriate non vanishing Christoffel symbols are carried out in this chapter. This is followed by the solution of the differential equations for the various spaces (namely, spaces of different curvatures).

Finally, in chapter five, we compare the physical interpretations of the solutions of the geodesic equations in 3-dimensional space time with those of 4-dimensional space time. We also discuss the deflection of a photon by a black hole.

Chapter 2

Mathematical tool for the project

2.1 Definitions of some common terminologies used in the project

The Universe

The universe contains everything. It consists of clusters of galaxies. From a cosmological point of view galaxies are the atoms of the universe and their distribution, motion and origin must be determined and explained. The Universe is a space that is homogeneous and isotropic.

Space

A space is defined by a set of rectangular coordinate system or axes.

Types of spaces to be discussed are **Euclidean** and **Riemannian** spaces

Euclidean space

Around 300BC the Greek mathematician Euclid undertook a study of relationships among distances and angles, first in a plane (2-dimensional space) and then in 3-dimensional space. An example of such a relationship is that the sum of angles in a triangle is always 180 degrees.

Euclidean structure

Euclidean space is a real coordinate space. In order to apply Euclidean geometry one needs to be able to talk about the distance between points and the angles between lines and vectors. The natural way to obtain these quantities is by introducing and using the standard inner product also known as dot product on R^n. The inner product of any two vectors **X** and **Y** is defined by

$$X \cdot Y = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

The result is always a real number. Furthermore, the inner product of **X** with itself is always non negative. This product allows us to define the length of a vector **X**

$$\|X\| = \sqrt{\vec{x}\cdot\vec{x}} = \sqrt{\sum_{i=1}^{n}(X_i)^2}$$

This length function satisfies the required properties of a norm and is called the Euclidean norm on R^n.

The angle $\theta\left(0 \le \theta \le 180\right)$ between **X** and **Y** is then given by

$$\theta = \cos^{-1}\left(\frac{\vec{x}\cdot\vec{y}}{\|X\|\|Y\|}\right)$$

Finally, one can use the norm to define a metric or distance function on R^n by

$$d\left(x,y\right) = \|x-y\| = \sqrt{\sum_{i=1}^{n}\left(x_i - y_i\right)^2} \text{..............................2.0}$$

This distance function is called the Euclidean metric.

Real coordinate space together with this Euclidean structure is called Euclidean space. Euclidean space is a space described by rectangular coordinates.

In three dimensional space the line element or the distance between two adjacent points (x, y, z) and $(x+dx, y+dy, z+dz)$ in Euclidean or Cartesian coordinates is given by

$$ds^2 = dx^2 + dy^2 + dz^2 \text{...2.1}$$

Riemannian space

Riemannian space is a space that cannot be covered with a set of rectangular coordinates. It is a space in which an element of distance can be defined. Riemannian space is endowed with a symmetric metric.

Space time

Let x^i (i=1, 2, 3, 4) be any functions of the ξ^i such that, to each set of values of the ξ^i there corresponds one set of values of x^i, and conversely. Where $x^i = x^i(\xi^1, \xi^2, \xi^3, \xi^4)$ then the x^i also will be accepted as coordinates, with respect to a new frame of reference, of the event whose coordinates were previously taken to be the ξ^i. It should be noted that, in general, each of the new coordinate's x^i will depend upon both time and the position of the event. It will not necessarily be the case that three of the coordinates x^i are spatial in nature and one is temporal. All possible events will now be mapped upon a space V_4, so that each event is represented by a point of the space and the x^i will be the coordinates of this point with respect to a coordinate frame. V_4 will be referred to as the **space-time continuum**.

Space time is therefore defined as space and time considered as one Point.

Examples of space-time are

1. 3–dimensional space time which consist of **two** spatial and **one** time component.
2. 4-dimensional space time which consist of **three** spatial and **one** time component.

The metric

A metric is a function of a topological space that gives, for any two points in the space, a value equal to the distance between them. In other words, an expression which expresses the distance between two adjacent points is called the metric or line element. In three dimensional Euclidean space, the line element or the distance between two adjacent points (x, y, z) and $(x+dx, y+dy, z+dz)$ in Cartesian coordinates is given by

$$ds^2 = dx^2 + dy^2 + dz^2 \text{...} 2.2$$

If we let $d\overline{s}$ be an element of distance in a different coordinate system such that $(\overline{x}^1, \overline{x}^2, \overline{x}^3)$ are the coordinates of a point in that system then the line element in that system is given by

$$d\overline{s}^2 = (d\overline{x}^1)^2 + (d\overline{x}^2)^2 + (d\overline{x}^3)^2$$

$$= \sum_{k=1}^{3} d\overline{x}^k d\overline{x}^k \text{..} 2.3$$

Since distance is invariant under coordinate transformation, we have

$$d\overline{s}^2 = ds^2 = \sum_{k=1}^{3} \frac{\partial \overline{x}^k}{\partial x^i} dx^i \frac{\partial \overline{x}^k}{\partial x^j} dx^j$$

$$= \sum_{k=1}^{3} \frac{\partial \overline{x}^k}{\partial x^i} \frac{\partial \overline{x}^k}{\partial x^j} dx^i dx^j$$

$$ds^2 = g_{ij}dx^i dx^j \text{...} 2.4$$

Where $$g_{ij} = \sum_{k=1}^{3} \frac{\partial \overline{x}^k}{\partial x^i} \frac{\partial \overline{x}^k}{\partial x^j} \text{..} 2.5$$

The quantities g_{ij} are collectively referred to as the metric tensor and equation (2.4) is called the **metric** or the **space-time metric** or the **fundamental metric** form. Examples of metrics:

1. In polar coordinates, the **contravariant** components of ds are $dx^1 = dr, dx^2 = rd\theta$ and we have

$$ds^2 = dr^2 + r^2 d\theta^2$$

2. In cylindrical coordinates ρ, φ, z ,the **contravariant** components of ds are $dx^1 = d\rho, dx^2 = \rho d\varphi, dx^3 = dz$ and we have

$$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2$$

3. In spherical coordinates r, θ, φ , the **contravariant** components are $dx^1 = dr, dx^2 = rd\theta, dx^3 = r\sin\theta d\varphi$, and we have

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$$

All of these concepts can be extended from three-dimensional space to four -dimensional spacetime without any difficulty. Using four-dimensional coordinates x^μ for describing the events and the world-line in spacetime, the separation of proper time or the separation between two events x^μ and $x^\mu + dx^\mu$ is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

For different coordinate systems, the dx^μ may not be the same, but the separation ds^2 remains unchanged. The metric tensor $g_{\mu\nu}$ determines the geometric character of the spacetime and has 16 components of which ten are independent since it is symmetric.

2.2 Tensor analysis

Physical laws must be independent of any particular coordinate systems used in describing them mathematically, if they are to be valid. A study of the consequences of this requirement leads to tensor analysis which is of great importance in general relativity theory, differential geometry, mechanics, hydrodynamics, electromagnetic theory and numerous fields of science and engineering.

In order to construct physical equations that are invariant under general coordinate transformation, we must know how the quantities described by the equations behave under these transformations. For some quantities, those defined directly in terms of coordinate differentials, the transformation properties may be determined by straight forward calculations. For other quantities, such as the electromagnetic fields, the transformation properties are partially a matter of definition. However, there is a tendency for all quantities of physical interest to transform in a reasonably simple way, for otherwise it would be difficult to put them together to form invariant equations. This section seeks to describe one class of objects whose transformation properties are particularly simple from quantities defined directly in terms of the coordinate system. The quantities or objects to be described are vectors, scalars, and tensors.

2.2.1 Scalars, Vectors and Tensors

Scalars

A scalar is a quantity that is invariant under coordinate transformation. The numerical value of a scalar at a point remains constant even if the coordinates of this point change. For example the interval ds between two points is a scalar. Also the scalar product of a covariant and contravariant vector is a scalar. If ϕ is a scalar field, then $\frac{\partial \phi}{\partial x^i}$ is a covariant vector because in a transformed coordinate system the gradient is

$$\frac{\partial \phi}{\partial \overline{x}^i} = \frac{\partial x^j}{\partial \overline{x}^i} \frac{\partial \phi}{\partial x^j} \qquad \text{.......} 2.6$$

Vectors

A vector is a tensor of rank 1

Two types of vectors are

Contravariant vectors

A vector A^i is said to be a contravariant vector if under the coordinate transformation

$x^i \to \bar{x}^i$, the A^i's transform according to the law

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j \quad \text{.........} 2.7$$

The rule of partial differentiation gives

$$dx^i = \frac{\partial x^i}{\partial x^j} dx^j \quad \text{.........} 2.8$$

So the coordinate differential is a contravariant vector

(ii) **Covariant vectors**

A vector A_i is said to be a covariant vector if under the coordinate transformation

$x^i \to \bar{x}^i$, the A_i's transform according to the law

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j \quad \text{.........} 2.9$$

Tensors

A tensor in an n-dimensional space is a set of n^r functions or quantities which transform between coordinate systems in a certain way. A tensor of zero rank is a single function of position, whose value is the same in all coordinate systems; in more familiar terms, it is a ***scalar.*** A tensor of rank one is a set of **n** functions; in more familiar terms, it is a ***vector,*** and the **n** functions are its components. A tensor of the second rank is a set of $\mathbf{n^2}$ functions, and the metric tensor g_{ij} is an example.

One great advantage of the tensor formalism is that, if physical laws are expressed as relations among tensors, they will be valid for any choice of coordinates.

2.2.2 Types of tensors

Contravariant tensor

The set of N^r quantities $T^{i_1 \cdots i_r}$ is said to constitute the components of a contravariant tensor of rank r at a point in an N-dimensional space if under the coordinate transformation $\overline{x}^j = \overline{x}^j\left(x^i\right)$ these quantities transform according to the law

$$\overline{T}^{i_1 \dots i_r} = \frac{\partial \overline{x}^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial \overline{x}^{i_r}}{\partial x^{j_r}} T^{j_1 \cdots j_r} \text{ ....2.10}$$

Covariant

The set of N^s quantities $T_{i_1 \cdots i_s}$ is said to constitute the components of a covariant tensor of rank s at a point P in an N- dimensional space if under the coordinate transformation $\overline{x}^j = \overline{x}^j(x^i)$, these quantities transform according to the law

$$\overline{T}_{i_1 \cdots i_s} = \frac{\partial x^{j_1}}{\partial \overline{x}^{i_1}} \cdots \frac{\partial x^{j_s}}{\partial \overline{x}^{i_s}} T_{j_1 \cdots j_s} \text{ ....2.11}$$

Mixed tensor

A set of N^{r+s} quantities $T^{i_1 \dots i_r}_{j_1 \cdots j_s}$ is said to constitute the components of a mixed tensor of rank r + s, contravariant of order or rank r and covariant of rank or order of s at a point P in an N – dimensional space if under the coordinate transformation $\overline{x}^j = \overline{x}^j(x^i)$ these quantities transform according to the law

$$\overline{T}^{i_1 \dots i_r}_{j_1 \dots j_s} = \frac{\partial \overline{x}^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial \overline{x}^{i_r}}{\partial x^{k_r}} \frac{\partial x^{l_1}}{\partial \overline{x}^{j_1}} \cdots \frac{\partial x^{l_s}}{\partial \overline{x}^{j_s}} T^{k_1 \dots k_r}_{l_1 \dots l_s} \text{ ....2.12}$$

(*David* Lovelock and Hanno Rund)[2]

2.2.3 Fundamental operations with tensors

Addition and Subtraction

The sum and differences of two or more tensors of the same rank and type are also tensors of the same rank and type. Thus if A_k^{ij} and B_k^{ij} are tensors then $C_k^{ij} = A_k^{ij} \pm B_k^{ij}$ are also tensors.

Proof

Writing the transformation laws for the given tensors, we have

$$\bar{A}_k^{ij} = \frac{\partial \bar{x}^i}{\partial x^r} \frac{\partial \bar{x}^j}{\partial x^s} \frac{\partial x^m}{\partial \bar{x}^k} A_m^{rs} \quad , \quad \bar{B}_k^{ij} = \frac{\partial \bar{x}^i}{\partial x^r} \frac{\partial \bar{x}^j}{\partial x^s} \frac{\partial x^m}{\partial \bar{x}^k} B_m^{rs} \quad \text{and}$$

$$\bar{A}_k^{ij} \pm \bar{B}_k^{ij} = \frac{\partial \bar{x}^i}{\partial x^r} \frac{\partial \bar{x}^j}{\partial x^s} \frac{\partial x^m}{\partial \bar{x}^k} \left(A_m^{rs} \pm B_m^{rs} \right), \text{ or, letting } C_k^{ij} = A_k^{ij} \pm B_k^{ij}$$

$$\bar{C}_k^{ij} = \frac{\partial \bar{x}^i}{\partial x^r} \frac{\partial \bar{x}^j}{\partial x^s} \frac{\partial x^m}{\partial \bar{x}^k} C_m^{rs} \quad \text{...2.13}$$

Outer multiplication

The product of two tensors is a tensor whose rank is the sum of the ranks of the given tensors.

Theorem

The outer product of two tensors of types (r_1, s_1) and (r_2, s_2) at a point P in x_n is a tensor of type $(r_1 + s_1, r_2 + s_2)$ at P.

Proof

Let R_{ij}^k and S^{lm} tensors, then under coordinate transformation $\bar{x}^j = \bar{x}^j \left(x^i \right)$ these tensors take the form

$$\bar{R}_{ij}^k = \frac{\partial \bar{x}^k}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial x^t}{\partial \bar{x}^j} R_{st}^r$$

$$\bar{S}^{lm} = \frac{\partial \bar{x}^l}{\partial x^p}\frac{\partial \bar{x}^m}{\partial x^q} S^{pq}$$

$$R_{ij}^k S^{lm} = \frac{\partial \bar{x}^k}{\partial x^r}\frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial x^t}{\partial \bar{x}^j}\frac{\partial \bar{x}^l}{\partial x^p}\frac{\partial \bar{x}^m}{\partial x^q} R_{st}^r S^{pq}$$ or letting $R_{st}^r S^{pq} = T_{st}^{rpq}$ we have

$$\bar{T}_{ij}^{klm} = \frac{\partial \bar{x}^k}{\partial x^r}\frac{\partial \bar{x}^l}{\partial x^p}\frac{\partial \bar{x}^m}{\partial x^q}\frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial x^t}{\partial \bar{x}^j} T_{st}^{rpq} \quad \text{...2.14}$$

which is the transformation law for a tensor of type $(3,2)$

The product which involves ordinary multiplication of the components of the tensor is called the outer product of the two tensors.

Contraction

If one contravariant and one covariant index of a tensor are set equal, the result indicates that a summation over the equal indices is to be taken according to the summation convention. This resulting sum is a tensor of rank two less than that of the original tensor. The process is called contraction. For example, in the tensor of rank 5, A_{qs}^{npr}, set $r = s$ to obtain $A_{qr}^{mpr} = B_q^{np}$ a tensor of rank 3.

The inner product

An inner product of two tensors of types (r_1, s_1) and (r_2, s_2) is a tensor of type ($r_1 + r_2 - 1, s_1 + s_2 - 1$) provided that the contraction is over a pair of indices one contra variant and the other covariant

Proof

Let T^{ij} be a contra variant tensor of the second rank, so that its transformation law is given by

$$\bar{T}^{ij} = \frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial \bar{x}^j}{\partial x^s} T^{rs}$$

And let C_P and F_q be two covariant vectors, with transformation laws

$$\bar{C}_p = \frac{\partial x^l}{\partial \bar{x}^p} C_l \quad \text{and} \quad \bar{F} = \frac{\partial x^m}{\partial \bar{x}^q} F_m$$

$$\overline{T}^{ij}\overline{C}_p\overline{F}_q = \frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial \overline{x}^j}{\partial x^s}\frac{\partial x^l}{\partial \overline{x}^p}\frac{\partial x^m}{\partial \overline{x}^q}T^{rs}C_lF_m \quad \text{or} \quad \text{if}$$

$Q^{ij}_{pq} = T^{ij}C_pF_q$, then

$$\overline{Q}^{ij}_{pq} = \frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial \overline{x}^j}{\partial x^s}\frac{\partial x^l}{\partial \overline{x}^p}\frac{\partial x^m}{\partial \overline{x}^q}Q^{rs}_{lm}$$

Contracting over the indices j and p we have

$$\overline{Q}^{ij}_{pq} = \frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial \overline{x}^j}{\partial x^s}\frac{\partial x^l}{\partial \overline{x}^j}\frac{\partial x^m}{\partial \overline{x}^q}Q^{rs}_{lm}$$

$$= \frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial x^m}{\partial \overline{x}^q}T^{rs}_{lm} \qquad \text{..2.15}$$

This is a transformation law for a tensor of type (1, 1). The contraction has reduced the both upper and lower indices by one. Such a product is called inner product.

The quotient theorem

If the product (outer or inner) of $R^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ with an arbitrary tensor yields a non- zero tensor of appropriate rank and character, then the quantities $R^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ are components of a tensor.

Example

Let us suppose that at a fixed point P of x_N, we are given a set of quantities a_i such that $a_i x^i$ is a scalar for any contravariant vector x^i at P such that

$a_i x^i = \phi$, where ϕ is a scalar

If we denote in $\overline{x}$ coordinate system

$$\overline{a}_j \overline{x}^j = \overline{\phi} = \phi \text{, then, we have}$$

$$a_i x^i = \phi = \overline{a}_j \overline{x}^j$$

$$a_i x^i = \overline{a}_j \frac{\partial \overline{x}^j}{\partial x^i} x^i$$

$$\left(a_i - \overline{a}_j \frac{\partial \overline{x}^j}{\partial x^i} \right) x^i = 0$$

But the x^i being the coordinates in the space x_N , are linearly independent we can therefore equate the coefficients to zero. Hence

$$a_i - \overline{a}_j \frac{\partial \overline{x}^j}{\partial x^i} = o$$

$a_i = \overline{a}_j \frac{\partial \overline{x}^j}{\partial x^i}$ or $\overline{a}_j = \frac{\partial x^i}{\partial \overline{x}^j} a_i$ shows that the a_i are the components of a covariant vector.

2.2.4 Symmetric and anti -symmetric tensors

Symmetric tensors

If two contravariant or covariant indices can be interchanged without altering the tensor, then the tenor is said to be symmetric with respect to these two indices. For example, if $A^{\mu\nu} = A^{\nu\mu}$ or $A_{\mu\nu} = A_{\nu\mu}$ then the contravariant tensor of second rank $A^{\mu\nu}$ or covariant tensor of second rank $A_{\mu\nu}$ is said to be symmetric.

For a tensor of higher rank $A_\lambda^{\mu\nu\sigma}$ if $A_\lambda^{\mu\nu\sigma} = A_\lambda^{\nu\mu\sigma}$ then the tensor $A_\lambda^{\mu\nu\sigma}$ is said to be symmetric with respect to the indices μ and ν

The symmetry property of a tensor is independent of the coordinate system used. So if a tensor is symmetric with respect to two indices in any coordinate system, it remains symmetric with respect to these two indices in any other coordinate system. This can be seen as follows,

If tensor $A_\lambda^{\mu\nu\sigma}$ is symmetric with respect to the indices μ and ν we have

$$A_\lambda^{\mu\nu\sigma} = A_\lambda^{\nu\mu\sigma}$$

According to tensor transformation law

$$\bar{A}_{\lambda}^{\mu\nu\sigma} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\lambda}} A_{\delta}^{\alpha\beta\gamma} = \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\lambda}} A_{\delta}^{\beta\alpha\gamma} = \bar{A}_{\lambda}^{\nu\mu\sigma}$$

Thus a given tensor is again symmetric with respect to first two indices in the new coordinate system. Similarly, this result can be proved for covariant indices.

Number of independent components of a symmetric tensor

A symmetric tensor of rank two in n-dimensional space has at most $\frac{n(n+1)}{2}$ independent components. This can be obtained as follows

The total number of components in the array is n^2, out of which all the n diagonal terms will in general be different and the rest ($n^2 - n$) will be equal in pairs. The number of pairs will be $\left(\frac{n^2 - n}{2}\right)$

Therefore the total number of independent components

$$= n + \left(\frac{n^2 - n}{2}\right) = \frac{n(n+1)}{2}$$

Antisymmetric tensors

A tensor whose components, change in sign but not in magnitude when two contravariant or covariant indices are interchanged, is said to be **anti-symmetric** or **skew symmetric** with respect to these two indices. For example, if

$A^{\mu\nu} = -A^{\nu\mu}$ or $A_{\mu\nu} = -A_{\nu\mu}$ then the contravariant tensor $A^{\mu\nu}$ or the covariant tensor $A_{\mu\nu}$ of second rank is anti- symmetric. For a tensor of higher rank $A_{\lambda}^{\mu\nu\sigma}$ if $A_{\lambda}^{\mu\nu\sigma} = -A_{\lambda}^{\nu\mu\sigma}$ then the tensor $A_{\lambda}^{\mu\nu\sigma}$ is anti-symmetric with respect to indices μ and ν

The skew symmetric property of a tensor is also independent of the choice of coordinate system. So, if a tensor is skew symmetric with respect to two indices in any coordinate system, it remains skew symmetric with respect to these two indices in any other coordinate system. To show this, let the tensor $A_{\lambda}^{\mu\nu\sigma}$ be antsymmetric with respect to the first two indices μ and ν, that is let

$$A_{\lambda}^{\mu\nu\sigma} = -A_{\lambda}^{\nu\mu\sigma} \ .$$

Then

$$\overline{A}_{\lambda}^{\mu\nu\sigma} = \frac{\partial \overline{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \overline{x}^{\nu}}{\partial x^{\beta}} \frac{\partial \overline{x}^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial \overline{x}^{\lambda}} A_{\delta}^{\alpha\beta\gamma} = -\frac{\partial \overline{x}^{\nu}}{\partial x^{\beta}} \frac{\partial \overline{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \overline{x}^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial \overline{x}^{\lambda}} A_{\delta}^{\beta\alpha\gamma} = -A_{\lambda}^{\nu\mu\delta}$$

The antisymmetry property, like the symmetry property, cannot be defined with respect to two indices of which one is contravariant and the other covariant.

If all the indices of a contravariant or covariant tensor can be interchanged so that its components change sign at each interchange of a pair of indices, then the tensor is said to be antisymmetric. That is if $A^{\mu\nu\sigma}$ are antisymmetric tensor then

$$A^{\mu\nu\sigma} = -A^{\nu\mu\sigma} = A^{\nu\sigma\mu}$$

Thus we may state that a contravariant or covariant tensor is antisymmetric if its components change sign under an odd permutation of its indices and do not change sign under an even permutation of its indices.

Number of independent components of an anti symmetric tensor

An antisymmetric tensor of rank two in n-dimensional space has $\frac{n(n-1)}{2}$ independent components. This can be shown as follows

The total number of components in the array is n^2, out of which all diagonal terms of the array will be zero since all the quantities $A^{\mu\mu}$ (no summation) are zero. The rest $(n^2 - n)$ will be pairwise equal in magnitude. The number of pairs will be $\frac{(n^2 - n)}{2}$. Therefore the total number of independent components $= \frac{n^2 - n}{2} = \frac{n(n-1)}{2}$

2.3 Covariant differentiation

2.3.1 Covariant differentiation of a vector A_i

By tensor transformation, we have

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j \quad \text{...2.16}$$

Differentiating with respect to $\bar{x}^k$, we have

$$\frac{\partial \bar{A}_i}{\partial \bar{x}^k} = \frac{\partial}{\partial \bar{x}^k}\left(\frac{\partial x^j}{\partial \bar{x}^i} A_j\right) = \frac{\partial x^j}{\partial \bar{x}^i}\frac{\partial A_j}{\partial x^r}\frac{\partial x^r}{\partial \bar{x}^k} + \frac{\partial^2 x^j}{\partial \bar{x}^i \partial \bar{x}^k} A_j \quad \text{........................ 2.17}$$

The presence of the second term on the right hand side shows that the partial derivatives $\frac{\partial \bar{A}_i}{\partial \bar{x}^k}$ do not transform like a tensor. This second term is called the affine term or the affine connection.

$\frac{\partial \bar{A}_i}{\partial \bar{x}^k} \neq 0$ even if $\frac{\partial A_j}{\partial x^r} = 0$ or A_j are constant

Transformation of the affine connection

Let the affine connection be defined by

$$\Gamma^m_{ik} = \frac{\partial x^m}{\partial x^j}\frac{\partial^2 x^j}{\partial x^i \partial x^k}$$

Passing form x^i to a different system $\bar{x}^i$ we find that

$$\bar{\Gamma}^m_{ik} = \frac{\partial \bar{x}^m}{\partial x^j}\frac{\partial^2 x^j}{\partial \bar{x}^k \partial \bar{x}^i}$$

$$= \frac{\partial \bar{x}^m}{\partial x^p}\frac{\partial x^p}{\partial x^j}\frac{\partial}{\partial \bar{x}^k}\left(\frac{\partial x^q}{\partial \bar{x}^i}\frac{\partial x^j}{\partial x^q}\right)$$

$$= \frac{\partial \bar{x}^m}{\partial x^p}\frac{\partial x^p}{\partial x^j}\left[\frac{\partial x^q}{\partial \bar{x}^i}\frac{\partial^2 x^j}{\partial \bar{x}^k \partial x^q} + \frac{\partial x^j}{\partial x^q}\frac{\partial^2 x^q}{\partial \bar{x}^k \partial \bar{x}^i}\right]$$

$$= \frac{\partial \overline{x}^m}{\partial x^p}\frac{\partial x^p}{\partial x^j}\left[\frac{\partial x^q}{\partial \overline{x}^i}\frac{\partial x^r}{\partial \overline{x}^k}\frac{\partial^2 x^j}{\partial x^r \partial x^q} + \frac{\partial x^j}{\partial x^q}\frac{\partial^2 x^q}{\partial \overline{x}^k \partial \overline{x}^i}\right]$$

$$= \frac{\partial \overline{x}^m}{\partial x^p}\frac{\partial x^p}{\partial x^j}\frac{\partial x^q}{\partial \overline{x}^i}\frac{\partial x^r}{\partial \overline{x}^k}\frac{\partial^2 x^j}{\partial x^r \partial x^q} + \frac{\partial \overline{x}^m}{\partial x^p}\frac{\partial x^p}{\partial x^j}\frac{\partial x^j}{\partial x^q}\frac{\partial^2 x^q}{\partial \overline{x}^k \partial \overline{x}^i}$$

$$= \frac{\partial \overline{x}^m}{\partial x^p}\frac{\partial x^q}{\partial \overline{x}^i}\frac{\partial x^r}{\partial \overline{x}^k}\frac{\partial x^p}{\partial x^j}\frac{\partial^2 x^j}{\partial x^r \partial x^q} + \frac{\partial \overline{x}^m}{\partial x^p}\frac{\partial^2 x^p}{\partial \overline{x}^k \partial \overline{x}^i}$$

$$\overline{\Gamma}^m_{ik} = \frac{\partial \overline{x}^m}{\partial x^p}\frac{\partial x^r}{\partial \overline{x}^k}\frac{\partial x^p}{\partial \overline{x}^i}\Gamma^p_{rq} + \frac{\partial \overline{x}^m}{\partial x^p}\frac{\partial^2 x^p}{\partial \overline{x}^k \partial \overline{x}^i} \quad \text{...} \ 2.18$$

From equation (2.16) and (2.18) we have

$$\overline{\Gamma}^m_{ik}\overline{A}_m = \left[\frac{\partial \overline{x}^m}{\partial x^p}\frac{\partial x^r}{\partial \overline{x}^k}\frac{\partial x^p}{\partial \overline{x}^i}\Gamma^p_{rq} + \frac{\partial \overline{x}^m}{\partial x^p}\frac{\partial^2 x^p}{\partial \overline{x}^k \partial \overline{x}^i}\right]\frac{\partial x^l}{\partial \overline{x}^m}A_l$$

$$\overline{\Gamma}^m_{ik}\overline{A}_m = \frac{\partial \overline{x}^m}{\partial x^p}\frac{\partial x^l}{\partial \overline{x}^m}\frac{\partial x^r}{\partial \overline{x}^k}\frac{\partial x^p}{\partial \overline{x}^i}\Gamma^p_{rq}A_l + \frac{\partial \overline{x}^m}{\partial x^p}\frac{\partial x^l}{\partial \overline{x}^m}\frac{\partial^2 x^p}{\partial \overline{x}^k \partial \overline{x}^i}A_l$$

$$\overline{\Gamma}^m_{ik}\overline{A}_m = \frac{\partial x^r}{\partial \overline{x}^k}\frac{\partial x^p}{\partial \overline{x}^i}\Gamma^p_{rq}A_p + \frac{\partial^2 x^p}{\partial \overline{x}^k \partial \overline{x}^i}A_p \quad \text{..} 2.19$$

Subtracting (2.19) from (2.17) leads to

$$\frac{\partial \overline{A}_i}{\partial \overline{x}^k} - \overline{\Gamma}^m_{ik}\overline{A}_m = \frac{\partial x^q}{\partial \overline{x}^i}\frac{\partial x^r}{\partial \overline{x}^k}\left[\frac{\partial A_q}{\partial x^r} - \Gamma^p_{rq}A_p\right] \quad \text{....................................} \ 2.20$$

We therefore define a covariant derivative of a covariant vector

$$A_{i;k} = \frac{\partial A_i}{\partial x^k} - \Gamma^m_{ik}A_m$$

Equation (2.20) tells us that $A_{i;k}$ is a tensor:

$$\overline{A}_{i;k} = \frac{\partial x^q}{\partial \overline{x}^i}\frac{\partial x^r}{\partial \overline{x}^k}\mathrm{A}_{q;r}$$

2.3.2 Covariant differentiation of a vector A^m

The transformation law for the vector A^m is given by

$$\bar{A}^m = \frac{\partial \bar{x}^m}{\partial x^i} A^i$$

Differentiating with respect to $\bar{x}^k$ gives

$$\frac{\partial \bar{A}^m}{\partial \bar{x}^k} = \frac{\partial}{\partial \bar{x}^k}\left(\frac{\partial \bar{x}^m}{\partial x^i} A^i\right)$$

$$\frac{\partial \bar{A}^m}{\partial \bar{x}^k} = \frac{\partial A^i}{\partial x^p}\frac{\partial x^p}{\partial \bar{x}^k}\frac{\partial \bar{x}^m}{\partial x^i} + \frac{\partial^2 \bar{x}^m}{\partial x^i \partial x^p}\frac{\partial x^p}{\partial \bar{x}^k} A^i \quad \text{.......................................2.21}$$

The first term on the right is what we would expect if $\frac{\partial A^m}{\partial x^k}$ were a tensor; the second term is what destroys the tensor behaviour.

Although $\frac{\partial A^m}{\partial x^k}$ is not a tensor, we can use to construct a tensor. Let us consider the identity

$$\frac{\partial \bar{x}^m}{\partial x^p}\frac{\partial x^p}{\partial \bar{x}^k} = \delta_k^m$$

Differentiating this identity with respect to $\bar{x}^i$, we have

$$\frac{\partial}{\partial \bar{x}^i}\left(\frac{\partial \bar{x}^m}{\partial x^p}\frac{\partial x^p}{\partial \bar{x}^k}\right) = 0$$

$$\frac{\partial \bar{x}^m}{\partial x^p}\frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^k} + \frac{\partial x^p}{\partial \bar{x}^k}\frac{\partial^2 \bar{x}^m}{\partial \bar{x}^i \partial x^p} = 0$$

$$\frac{\partial \bar{x}^m}{\partial x^p}\frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^k} = -\frac{\partial x^p}{\partial \bar{x}^k}\frac{\partial x^q}{\partial \bar{x}^i}\frac{\partial^2 \bar{x}^m}{\partial x^q \partial x^p}$$

We therefore write equation **2.18** as

$$\overline{\Gamma}^m_{ik} = \frac{\partial \overline{x}^m}{\partial x^p}\frac{\partial x^r}{\partial \overline{x}^k}\frac{\partial x^q}{\partial \overline{x}^i}\Gamma^p_{rq} - \frac{\partial x^p}{\partial \overline{x}^k}\frac{\partial x^q}{\partial \overline{x}^i}\frac{\partial^2 \overline{x}^m}{\partial x^q \partial x^p}$$

From this equation, we see that

$$\overline{\Gamma}^m_{ik}\overline{A}^i = \frac{\partial \overline{x}^m}{\partial x^p}\frac{\partial x^r}{\partial \overline{x}^k}\frac{\partial x^q}{\partial \overline{x}^i}\frac{\partial \overline{x}^i}{\partial x^s}\Gamma^p_{rq}A^s - \frac{\partial x^p}{\partial \overline{x}^k}\frac{\partial x^q}{\partial \overline{x}^i}\frac{\partial \overline{x}^i}{\partial x^s}\frac{\partial^2 \overline{x}^m}{\partial x^q \partial x^p}A^s$$

$$\overline{\Gamma}^m_{ik}\overline{A}^i = \frac{\partial \overline{x}^m}{\partial x^p}\frac{\partial x^r}{\partial \overline{x}^k}\Gamma^p_{rq}A^q - \frac{\partial x^p}{\partial \overline{x}^k}\frac{\partial^2 \overline{x}^m}{\partial x^q \partial x^p}A^q$$

Adding this equation to equation **2.21**, and rearranging the dummy indices we have

$$\frac{\partial \overline{A}^m}{\partial \overline{x}^k} + \overline{\Gamma}^m_{ki}\overline{A}^i = \frac{\partial A^q}{\partial x^p}\frac{\partial x^p}{\partial \overline{x}^k}\frac{\partial \overline{x}^m}{\partial x^q} + \frac{\partial \overline{x}^m}{\partial x^p}\frac{\partial x^r}{\partial \overline{x}^k}\Gamma^p_{rq}A^q$$

$$\frac{\partial \overline{A}^m}{\partial \overline{x}^k} + \overline{\Gamma}^m_{ki}\overline{A}^i = \frac{\partial x^p}{\partial \overline{x}^k}\frac{\partial \overline{x}^m}{\partial x^q}\left[\frac{\partial A^q}{\partial x^p} + \Gamma^q_{pr}A^r\right]$$

We therefore define a covariant derivative of a cotravariant vector

$$A^m_{;k} = \frac{\partial A^m}{\partial x^k} + \Gamma^m_{ki}A^i$$

2.4 The relation of Christoffel symbols to the metric tensor

In the computation of covariant derivative of a tensor in Euclidean space demands as a pre-requisite the evaluation of affine connections

$$\Gamma_{ik}^{m} = \frac{\partial^2 x^j}{\partial x^i \partial x^k} \frac{\partial x^m}{\partial x^j} \quad \text{...} 2.22$$

This formula shows that $\Gamma_{ik}^{m} = \Gamma_{ki}^{m}$ that is affine connection is symmetric in its subscript but this does not make the most convenient way of calculating the coefficient of Γ_{ik}^{m}. It turns out that we can relate Γ_{ik}^{m} directly to the metric tensor and that the resulting formula is easier to deal with.

We show that the covariant derivative of the metric tensor g_{ik} is zero

Using the relation

$$\delta A_i = g_{ik}\, \delta A^k \quad \text{...} 2.23$$

This is valid for the vector ∂A_i and also for any vector

On the other hand,

$A_i = g_{ik} A^k$ So that

$$\delta A_i = \delta\left(g_{ik} A^k\right) = A^k \delta g_{ik} + g_{ik}\, \delta A^k \quad \text{......................................} 2.24$$

Substituting (2.23) into (2.24)

We have

$$A^k \delta g_{ik} = 0$$

And since the vector A^k is arbitrary, we have

$$\delta A_{ik} = g_{ik:l} = 0 \quad \text{..} 2.25$$

This g_{ik} may be considered to be a constant during covariant differentiation

Now, writing out (2.25) explicitly, we obtain

$$g_{ij:k} = \frac{\partial g_{ij}}{\partial x^k} - \Gamma^n_{ik} g_{nj} - \Gamma^n_{jk} g_{in} = 0 \text{...} 2.26$$

Performing cyclic permutation of the indices i, j, k we obtain two other formulae

$$g_{jk:i} = \frac{\partial g_{jk}}{\partial x^i} - \Gamma^n_{ji} g_{nk} - \Gamma^n_{ki} g_{jn} = 0 \text{...} 2.27$$

$$g_{kl:j} = \frac{\partial g_{kl}}{\partial x^j} - \Gamma^n_{kj} g_{ni} - \Gamma^n_{ij} g_{kn} = 0 \text{...} 2.28$$

Adding (2.27) and (2.28) and subtracting (2.26), on using the symmetric properties of g_{ij} and Γ^n_{ik} we have

$$\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} - 2\Gamma^n_{ij} g_{kn} = 0$$

$$2 g_{nk} \Gamma^n_{ij} = \frac{\partial g_{ik}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \text{...} 2.29$$

Multiply both sides of (2.29) by $\frac{1}{2} g^{nk}$ and using the fact that $g^{nk} g_{mk} = \delta^n_m$

We have

$$\Gamma^n_{ij} = \frac{1}{2} g^{nk} \left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right] \text{...} 2.30$$

$$\Gamma^i{}_{ij;k} = \frac{1}{2} \left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right] \text{..} 2.31$$

Equation (2.30) is called the **chriotoffel symbol of the second kind** and (2.31) is called Christoffel symbol of the first kind

The relationship between the two Christoffel symbols is

$$\begin{bmatrix} n \\ ij \end{bmatrix} = g^{nk}[i \quad j \quad ;k]$$

From (2.30)

Put $n = i$

$$\Gamma_{ij}^{i} = \frac{1}{2} g^{ik} \left[\frac{\partial g_{jk}}{\partial x^{i}} + \frac{\partial g_{ki}}{\partial x^{j}} - \frac{\partial g_{ij}}{\partial x^{k}} \right] \text{..} 2.32$$

$\Gamma_{j\ \ k}^{\ i} = 0$ for i ≠ j ≠ k

Put $k = i$, in equation (2.32) we have

$$\Gamma_{ij}^{i} = \frac{1}{2} g^{ii} \left[\frac{\partial g_{ji}}{\partial x^{i}} + \frac{\partial g_{ii}}{\partial x^{j}} - \frac{\partial g_{ij}}{\partial x^{i}} \right]$$

$$\Gamma_{ij}^{i} = \frac{1}{2} g^{ii} \frac{\partial g_{ii}}{\partial x^{j}} \text{ Since } g^{ii} g_{ii} = 1$$

$$\Gamma_{ij}^{i} = \frac{1}{2g_{ii}} \frac{\partial g_{ii}}{\partial x^{j}} \text{..} 2.33$$

Put $j = i$ in (2.32), we have

$$\Gamma_{ii}^{i} = \frac{1}{2} g^{ik} \left[\frac{\partial g_{ik}}{\partial x^{i}} + \frac{\partial g_{ki}}{\partial x^{i}} - \frac{\partial g_{ii}}{\partial x^{k}} \right]$$

Put $k = i$

$$\Gamma_{ii}^{i} = \frac{1}{2} g^{ii} \left[\frac{\partial g_{ii}}{\partial x^{i}} + \frac{\partial g_{ii}}{\partial x^{i}} - \frac{\partial g_{ii}}{\partial x^{i}} \right]$$

$$\Gamma_{ii}^{i} = \frac{1}{2g_{ii}} \frac{\partial g_{ii}}{\partial x^{i}} \text{..} 2.34$$

From (2.32) put $i = j$

$$\Gamma_{jj}^{n} = \frac{1}{2} g^{nk} \left[\frac{\partial g_{jk}}{\partial x^{j}} + \frac{\partial g_{kj}}{\partial x^{j}} - \frac{\partial g_{jj}}{\partial x^{k}} \right]$$

Put $n = i$

$$\Gamma^i_{jj} = \frac{1}{2} g^{ik} \left[\frac{\partial g_{jk}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^j} - \frac{\partial g_{jj}}{\partial x^k} \right]$$

Put $k = i$

$$\Gamma^i_{jj} = \frac{1}{2} g^{ii} \left[\frac{\partial g_{ji}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^j} - \frac{\partial g_{jj}}{\partial x^i} \right]$$

$$\Gamma^i_{jj} = \frac{1}{2} g^{ii} \left(\frac{-\partial g_{jj}}{\partial x^i} \right) \text{ Since } \frac{\partial g_{ji}}{\partial x^j} = \frac{\partial g_{ij}}{\partial x^j} = 0$$

Hence $$\Gamma^i_{jj} = \frac{1}{-2g_{ii}} \frac{\partial g_{jj}}{\partial x^i} \quad \text{..2.35}$$

Chapter 3

Geodesics

Let M be an n-dimensional differentiable manifold, upon which the usual structure of vectors, tensors, and differential forms has been defined.

A ***geodesic*** on M is a curve whose tangent vector is parallel transported, that is $u^i = \frac{dx^i}{d\lambda}$

A vector u^i is said to be parallel transported along a curve $x^i = x^i(\lambda)$ if it satisfies the equation

$$\frac{du^i}{d\lambda} + \Gamma^i_{kl}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda} = 0$$

where Γ^i_{kl} is the affine connection

A ***geodesic*** is also a curve of extremal arc length.

As examples, the geodesics on a plane are straight lines where as the geodesics on a sphere are arcs of great circles.

3.1 Geodesics in Euclidean plane

Since finding geodesic in Euclidean space involves finding the minimum curve or finding the shortest path between two points, we therefore apply calculus of variation which will lead us to the **Euler-Lagrange differential equation.** This equation is derived as follows

We consider a function $f\left(y,\dot{y},x\right)$ defined on a path $y=y(x)$ between two points $\left(x_1,y_1\right)$ and $\left(x_2,y_2\right)$. We wish to find a particular path $y\left(x\right)$ such that the line integral J of the function f between x_1 and x_2

$$J=\int_{x_1}^{x_2} f\left(y,\dot{y},x\right)\mathrm{dx} \text{ ....3.0}$$

has a stationary value relative to paths differing infinitesimally from the correct function $y\left(x\right)$. The variable x here plays the role of a parameter and we consider only such varied paths for which $y(x_1)=y_1$, $y(x_2)=y_2$.**(Figure 3.1)**

Where $\dot{y}=\dfrac{dy}{dx}$

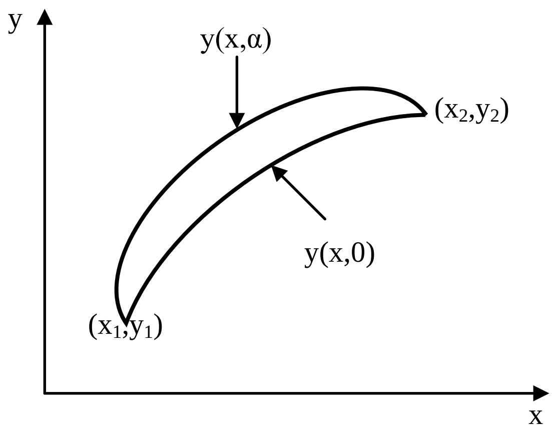

Figure 3.1 Varied paths in the one-dimensional extremum problem

If $y\left(x\right)$ is the path and $y\left(x,\alpha\right)$ is the path of variation then, the combination of the two gives the fixed space.

$$y(x,\alpha) = y(x,0) + \alpha\eta(x) \text{ ....3.1}$$

Where $\eta(x) = \left(\frac{\partial y}{\partial \alpha}\right)_{\alpha=0}$ is assumed to be continuously differentiable in the open interval $[x_1, x_2]$ and such that $\eta(x_1) = \eta(x_2) = 0$ then J is also a function of α . Hence

$$J(\alpha) = \int_{x_1}^{x_2} f\left[y(x,\alpha), \dot{y}(x,\alpha), x\right]dx \text{ ....3.2}$$

The condition for obtaining an extremum is given by

$$\left(\frac{\partial J}{\partial \alpha}\right)_{\alpha=0} = 0 \text{ ....3.3}$$

Thus differentiating (3.2) we obtain

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\right]dx \text{ ....3.4}$$

The second integral of (3.4) is

$$\int_{x_1}^{x_2}\frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}dx = \int_{x_1}^{x_2}\frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial x \partial \alpha}dx = \left(\frac{\partial f}{\partial \dot{y}}\frac{\partial y}{\partial \alpha}\right)_{x_1}^{x_2} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\frac{\partial y}{\partial \alpha}dx$$

The condition of all varied curves is that they pass through the points (x_1, y_1) and (x_2, y_2) and hence $\frac{dy}{d\alpha}$ must vanish at those points. Equation (3.4) becomes

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\frac{\partial y}{\partial \alpha}\right]dx$$

$$= \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\right]\frac{\partial y}{\partial \alpha}dx$$

Multiplying through by $d\alpha$ and evaluate the derivative at $\alpha = 0$, we have

$$\partial J = \left(\frac{\partial J}{\partial \alpha}\right)d\alpha = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\dot{y}}\right)\right]\left(\frac{\partial y}{\partial \alpha}\right)d\alpha dx = 0$$

$$\partial J = \left(\frac{\partial J}{\partial \alpha}\right) d\alpha = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\dot{y}}\right)\right] \delta y dx = 0$$

Where $\delta y = \left(\frac{\partial y}{\partial \alpha}\right)_0 d\alpha$

$$\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\right] \delta y = 0$$

The quantity $\delta y = \frac{\partial y}{\partial \alpha} d\alpha$ is known as the variation of y. It is an arbitrary quantity and therefore generally not equal to zero. Hence we must have

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0 \text{ ....3.5}$$

Thus J is an extremum only for curves $y(x)$ such that $f\left(y, \dot{y}, x\right)$ satisfies the differential equation (3.5). This differential equation is the usual Lagrange equation in classical mechanics. In calculus of variation, it is known as the Euler-Lagrange equation.

[*Herbert Goldstein*][3]

Illustration

We prove that the shortest distance between two points in a plane or Euclidean space is a straight line.

Let the line element or an element of distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ be given by

$$ds^2 = dx^2 + dy^2 \text{ ....3.6}$$

$$ds = [1 + (\frac{dy}{dx})^2]^{\frac{1}{2}} dx$$

$$ds = (1 + \dot{y}^2)^{\frac{1}{2}} dx \text{ ....3.7}$$

Where $\dot{y} = \frac{dy}{dx}$

Using Hamilton's principle, the total distance between these two points is

$$L = \int_{x_1}^{x_2} (1+\dot{y}^2)^{\frac{1}{2}} dx = \int_{x_1}^{x_2} f dx \quad ....3.8$$

Where $\qquad f = (1+\dot{y}^2)^{\frac{1}{2}}$ 3.9

This does not explicitly depend on y so using the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$$

But $\frac{\partial f}{\partial y} = 0$ therefore $\qquad \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$

From equation (3.9) $\qquad \frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{(1+\dot{y}^2)^{\frac{1}{2}}}$

Hence $\qquad \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = \frac{d}{dx}\left(\frac{\dot{y}}{(1+\dot{y}^2)^{\frac{1}{2}}}\right) = 0$

$$\frac{\dot{y}}{(1+\dot{y}^2)^{\frac{1}{2}}} = C$$

Where C is a constant of integration from which, we have

$$\dot{y} = C(1+\dot{y}^2)^{\frac{1}{2}}$$

$$\dot{y}^2 = C^2(1+\dot{y}^2)$$

$$\dot{y}^2 = \frac{C^2}{1-C^2}$$

$$\dot{y} = \frac{C}{\sqrt{1-C^2}} = a \quad \text{(Constant)}$$

$$\dot{y} = a$$

$$\frac{dy}{dx} = a$$

$$y = ax + b$$

Where b is a constant of integration

Hence, the geodesic in Euclidean plane is a straight line

3.2 General geodesic equations in Non-Euclidean or Riemannian space

Two properties of a straight line can be generalised as

i.) The property of straightness.

ii) The property of shortest distance between two points

(*Jayant* V. Narliker)[1]

These **two** properties are used to formulate the general geodesic equation in Riemannian space.

1. **Straightness means that as we move along the line, its direction does not change**. Generalization of this concept;

Let $x^i(\lambda)$ be the parametric representation of a curve in spacetime. The tangent vector for this parametric representation is

$$u^i = \frac{dx^i}{d\lambda} \quad \text{..3.10}$$

The straightness criterion demands that u^i should not change along a curve

On moving from λ to $\lambda + \delta\lambda$ in a curve, the change u^i is given by

$$\Delta u^i = du^i - \delta u^i$$

But $$\delta u^i = -\Gamma^i_{kl} u^k \delta x^l$$

$$\Delta u^i = \frac{du^i}{d\lambda}\delta\lambda + \Gamma^i_{kl} u^k \delta x^l$$

$$\Delta u^i = \frac{du^i}{d\lambda}\delta\lambda + \Gamma^i_{kl^i} u^k \delta x^l \text{.......................................3.11}$$

The second term on the right hand side of (3.11) arises from the change produced by parallel transport through coordinate displacement δx^l but

$$\delta x^l = u^l \delta\lambda \quad ....3.12$$

Substituting (3.12) into (3.11), we have

$$\Delta u^i = \frac{du^i}{d\lambda}\delta\lambda + \Gamma^i_{kl} u^k u^l \delta\lambda$$

$$\Delta u^i = \left[\frac{du^i}{d\lambda} + \Gamma^i_{kl} u^k u^l\right]\delta\lambda \quad ....3.13$$

The condition for no change of direction u^i implies $\Delta u^i = 0$

Equation (3.13) becomes

$$\frac{du^i}{d\lambda} + \Gamma^i_{kl} u^k u^l = 0 \qquad \text{Or}$$

$$\frac{d^2 x^i}{d\lambda^2} + \Gamma^i_{kl}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda} = 0 \quad ....3.14$$

Equation (3.14) is the condition that our curve must satisfy in order to be straight.

2. **Property of shortest distance between two points**

Let a curve parameterised by λ connects two points P_1 and P_2 of spacetime with parameters λ_1 and λ_2 respectively. Then the distance of P_2 from P_1 is defined as

$$S(P_2, P_1) = \int_{\lambda_1}^{\lambda_2}\left[g_{ik}\frac{dx^i}{d\lambda}\frac{dx^k}{d\lambda}\right]^{\frac{1}{2}} d\lambda = \int_{\lambda_1}^{\lambda_2} L\, d\lambda \quad ....3.15$$

Where $$L = \left(g_{ik}\frac{dx^i}{d\lambda}\frac{dx^k}{d\lambda}\right)^{\frac{1}{2}}$$

For small displacement of a curve connecting P_1 and P_2, we demand that $S(P_2, P_1)$ be stationary and these displacement vanish at P_1 and P_2. This is a standard problem in the calculus of variation and its solution leads to the familiar Euler-Lagrange equation;

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^j}\right) - \frac{\partial L}{\partial x^j} = 0 \quad ....3.16$$

Where j= i, k

But $\quad L=\left(g_{ik}\frac{dx^i}{d\lambda}\frac{dx^k}{d\lambda}\right)^{\frac{1}{2}}=(g_{ik}\dot{x}^i\dot{x}^k)^{\frac{1}{2}}$

$$\frac{\partial L}{\partial \dot{x}^j}=\frac{1}{2}g_{ik}\left(\dot{x}^k+\dot{x}^i\right)\left(g_{ik}\dot{x}^i\dot{x}^k\right)^{-\frac{1}{2}}$$

$$=\frac{1}{2L}g_{ik}\left(\dot{x}^k+\dot{x}^i\right)$$

$$\frac{\partial L}{\partial x^j}=\frac{1}{2}\frac{\partial g_{mn}}{\partial x^j}\frac{dx^m}{d\lambda}\frac{dx^n}{d\lambda}\left(g_{ik}\frac{dx^i}{d\lambda}\frac{dx^k}{d\lambda}\right)^{\frac{-1}{2}}$$

$$\frac{\partial L}{\partial x^j}=\frac{1}{2L}\frac{\partial g_{mn}}{\partial x^j}\frac{dx^m}{d\lambda}\frac{dx^n}{d\lambda}$$

Substituting these equations in to (3.16), we have

$$\frac{d}{d\lambda}\left(\frac{1}{2L}g_{ik}\left(\dot{x}^k+\dot{x}^i\right)\right)-\frac{1}{2L}\frac{\partial g_{mn}}{\partial x^j}\frac{dx^m}{d\lambda}\frac{dx^n}{d\lambda}=0$$

$$\frac{d}{d\lambda}\left(g_{ik}\left(\dot{x}^k+\dot{x}^i\right)\right)-\frac{\partial g_{mn}}{\partial x^j}\frac{dx^m}{d\lambda}\frac{dx^n}{d\lambda}=0 \text{...3.17}$$

Using $ds=Ld\lambda$ equation (3.17) becomes

$$L^2\frac{d}{ds}\left(g_{ik}\frac{dx^k}{ds}+g_{ik}\frac{dx^i}{ds}\right)-L^2\frac{\partial g_{mn}}{\partial x^j}\frac{dx^m}{ds}\frac{dx^n}{ds}=0 \text{ ....3.18}$$

Changing the dummy indices in the first term of (3.18)

$$\frac{d}{ds}\left(g_{mk}\frac{dx^m}{ds}+g_{kn}\frac{dx^n}{ds}\right)-\frac{\partial g_{mn}}{\partial x^j}\frac{dx^m}{ds}\frac{dx^n}{ds}=0$$

$$g_{mk}\frac{d^2x^m}{ds^2}+\frac{dx^m}{ds}\frac{dg_{mk}}{ds}+g_{kn}\frac{d^2x^n}{ds^2}+\frac{dx^n}{ds}\frac{dg_{kn}}{ds}-\frac{\partial g_{mn}}{\partial x^k}\frac{dx^m}{ds}\frac{dx^n}{ds}=0 \text{ ....3.19}$$

But $\quad \frac{dg_{mk}}{ds}=\frac{\partial g_{mk}}{\partial x^n}\frac{dx^n}{ds}$ and $\frac{dg_{kn}}{ds}=\frac{\partial g_{kn}}{\partial x^m}\frac{dx^m}{ds}$

Substituting these equations into (3.19) we have

$$g_{mk}\frac{d^2x^m}{ds^2}+\frac{dx^m}{ds}\frac{\partial g_{mk}}{\partial x^n}\frac{dx^n}{ds}+g_{kn}\frac{d^2x^n}{ds^2}+\frac{dx^n}{ds}\frac{\partial g_{kn}}{\partial x^m}\frac{dx^m}{ds}-\frac{\partial g_{mn}}{\partial x^k}\frac{dx^m}{ds}\frac{dx^n}{ds}=0$$

$$g_{mk}\frac{d^2x^m}{ds^2}+g_{kn}\frac{d^2x^n}{ds^2}+\left(\frac{\partial g_{mk}}{\partial x^n}+\frac{\partial g_{kn}}{\partial x^m}-\frac{\partial g_{mn}}{\partial x^k}\right)\frac{dx^m}{ds}\frac{dx^n}{ds}=0 \quad \text{..........................3.20}$$

Setting m, n = i in the first two terms of (3.20), we have

$$g_{ik}\frac{d^2x^i}{ds^2}+g_{ki}\frac{d^2x^i}{ds^2}+\left(\frac{\partial g_{mk}}{\partial x^n}+\frac{\partial g_{kn}}{\partial x^m}-\frac{\partial g_{mn}}{dx^k}\right)\frac{dx^m}{ds}\frac{dx^n}{ds}=0$$

$$2g_{ik}\frac{d^2x^i}{ds^2}+\left(\frac{\partial g_{mk}}{\partial x^n}+\frac{\partial g_{kn}}{\partial x^m}-\frac{\partial g_{mn}}{\partial x^k}\right)\frac{dx^m}{ds}\frac{dx^n}{ds}=0$$

$$\frac{d^2x^i}{ds^2}+\frac{1}{2}g^{ik}\left(\frac{\partial g_{mk}}{\partial x^n}+\frac{\partial g_{kn}}{\partial x^m}-\frac{\partial g_{mn}}{\partial x^k}\right)\frac{dx^m}{ds}\frac{dx^n}{ds}=0$$

Hence

$$\frac{d^2x^i}{ds^2}+\Gamma^i_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}=0 \quad \text{...3.21}$$

It is clear that equations (3.14) and (3.21) are the same although S in (3.21) has the special meaning of length along the curve while λ in (3.14) appears to be general. If (3.14) is satisfied then λ must be a constant multiple of S.

The N equations (3.21) are second-order differential equations for the functions $x^i(s)$ and their solutions will involve 2N arbitrary constants. If these equations are satisfied at every point of the curve $x^i(s)$, it is a **geodesic**. If A, B are two given points having coordinates $x^i = a^i, x^i = b^i$ respectively, the 2N conditions that the geodesic must contain these points will, in general determine the arbitrary constants. Hence there is, in general a unique geodesic connecting every pair of points. However, in some cases this will never be so. For instance, the geodesics on the surface of a sphere (R_2) are great circles and in general there are two great circle arcs joining two given points, major arc and a minor arc (**figure 3.2**)

(D.F Lawden) [4].

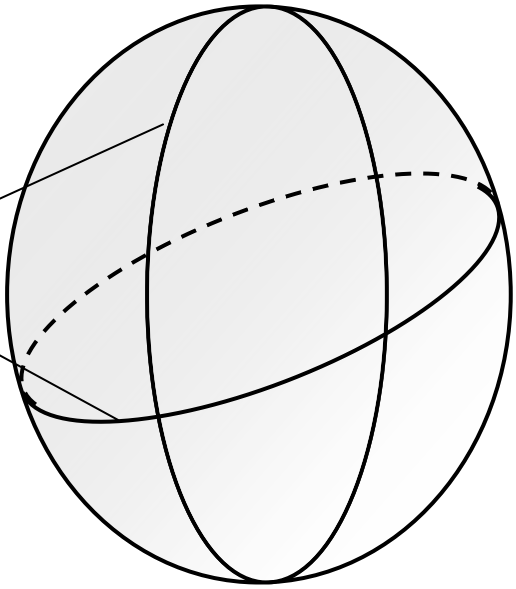

Figure 3.2 Diagram showing the great circles of a spherical object.

3.2.1 To show that the geodesic equation has a first integral

From $L = \left(g_{ik}\dot{x}^i\dot{x}^k\right)^{\frac{1}{2}}$ since L does not explicitly depend on λ ,we have

$$L - \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} = c \qquad \text{...3.22}$$

Where c is a constant

But $$\frac{\partial L}{\partial \dot{x}^i} = \frac{1}{2} g_{ik}\dot{x}^k \left(g_{ik}\dot{x}^i\dot{x}^k\right)^{\frac{-1}{2}} = \frac{1}{2L} g_{ik}\dot{x}^k$$

$$L - \dot{x}^i \frac{1}{2L} g_{ik}\dot{x}^k = c$$

$$L - \frac{1}{2L} g_{ik}\dot{x}^i\dot{x}^k = c$$

$$L - \frac{1}{2L} L^2 = c$$

$$\frac{L}{2} = c$$

$$L^2 = 4c^2 = cons\tan t \qquad g_{ik}\dot{x}^i\dot{x}^k = cons\tan t$$

Hence

$$g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}=C \quad \text{...3.23}$$

Where C is the constant of integration

Characteristics of equation (3.23) are;

If g_{ij} has one positive eigenvalue and three negative eigenvalues, then

(i) $C = 0$ for a null geodesic

(ii.) $C > 0$ for time like curves

(iii). $C < 0$ for space like curves

The λ is called an affine parameter

3.3 The curvature tensor

One property that distinguishes a curved space from a flat space is the curvature of the space. The curvature of the space is defined by the curvature tensor which is derived as follows;

Let a vector x^j at a point **P** undergo parallel transport to another point **Q.**

The component of the x^j will change, the covariant differentiation of the vector x^j is given by

$$x^j{}_{;h} = \frac{\partial x^j}{\partial x^h} + \Gamma^j{}_{lh} x^l \quad \text{...3.24}$$

Taking the second derivative of the above equation, we have

$$x^j{}_{;h;k} = \frac{\partial}{\partial x^k}\left(\frac{\partial x^j}{\partial x^h} + \Gamma^j{}_{lh} x^l\right)$$

$$= \frac{\partial (x^j{}_{;h})}{\partial x^k} + \Gamma^j{}_{mk} x^m{}_{;h} - \Gamma_{hk}{}^l x^j{}_{;l}$$

$$x^j{}_{;h;k} = \frac{\partial}{\partial x^k}\left[\frac{\partial x^j}{\partial x^h} + \Gamma^j{}_{lh} x^l\right] + \Gamma^j{}_{mk}\left[\frac{\partial x^m}{\partial x^h} + \Gamma^m{}_{lh} x^l\right] - \Gamma^l{}_{hk} x^j{}_{;l}$$

$$= \frac{\partial^2 x^j}{\partial x^k \partial x^h} + \frac{\partial \Gamma_{lh}{}^j}{\partial x^k} x^l + \Gamma^j_{lh} \frac{\partial x^l}{\partial x^k} + \Gamma_{mk}{}^j \frac{\partial x^m}{\partial x^h} + \Gamma_{mk}{}^j \Gamma_{lh}{}^m x^l - \Gamma_{hk}{}^l x_{;l}{}^j \quad \text{...................3.25}$$

Interchanging h and k, we have

$$x^j{}_{;k;h} = \frac{\partial^2 x^j}{\partial x^h \partial x^k} + \frac{\partial \Gamma_{lk}{}^j}{x^h} x^l + \Gamma^j_{lh} \frac{\partial x^l}{\partial x^k} + \Gamma_{mh}{}^j \frac{\partial x^m}{\partial x^k} + \Gamma^j{}_{mh} \Gamma^m{}_{lk} x^l - \Gamma^l{}_{kh} x^j{}_{;l} \quad \text{.......................3.26}$$

Subtracting equation (3.26) from (3.25) and considering the fact that $\frac{\partial^2 x^j}{\partial x^h \partial x^k}$ are symmetric that is

$$\frac{\partial^2 x^j}{\partial x^h \partial x^k} = \frac{\partial^2 x^j}{\partial x^k \partial x^h}, \quad \text{we have}$$

$$x^j{}_{;h;k} - x^j{}_{;k;h} = \left[\frac{\partial \Gamma^j{}_{lh}}{\partial x^k} - \frac{\partial \Gamma^j lk}{\partial x^h} + \Gamma^j{}_{mk}\Gamma^m{}_{lh} - \Gamma^j{}_{mh}\Gamma^m{}_{lk}\right] x^l - \left[\Gamma^l{}_{hk} - \Gamma^l{}_{kh}\right] x^j{}_{;l} \quad \text{.............3.27}$$

Where

$$K^{J}{}_{lhk} = \frac{\partial \Gamma^{j}{}_{lh}}{\partial x^{k}} - \frac{\partial \Gamma^{j}{}_{lk}}{\partial x^{h}} + \Gamma^{j}{}_{mk}\Gamma^{m}{}_{lh} - \Gamma^{j}{}_{mh}\Gamma^{m}{}_{lk} \quad \text{.................................3.28}$$

And

$$S^{l}{}_{hk} = \Gamma^{l}{}_{hk} - \Gamma^{l}{}_{kh} \quad \text{..3.29}$$

The quantity $K^{j}{}_{lhk}$ is called the curvature tensor or Riemannian tensor of type (1, 3). This can be seen by using the symmetric property of the affine connections $\Gamma^{l}{}_{hk} = \Gamma^{l}{}_{kh}$ in equation (3.27) we obtain

$$X^{j}{}_{;h;k} - X^{j}{}_{;k;h} = K^{j}{}_{lhk} x^{l} \quad \text{..3.30}$$

The quantity on the left hand side of equation (3.30) is a component of a third rank tensor since it is the difference between the components of two tensors. Therefore the quantity on the right is also a component of a third rank tensor. It has the form of a contracted product of a fourth rank tensor whose components are $K^{j}{}_{lhk}$ and the first rank tensor whose components are x^{l}. We may infer that $K^{j}{}_{lhk}$ are components of a fourth rank tensor called the Riemann–christoffel curvature tensor. It is usually denoted by $R^{j}{}_{lhk}$.

The quantity $S^{l}{}_{hk}$ is the component of a type (1, 2) tensor called the torsion tensor.

The space endowed with symmetric affine connections, that is $\Gamma^{l}{}_{hk} = \Gamma^{l}{}_{kh}$, the torsion tensor vanishes identically but the curvature tensor does not in general vanish.

Space-time is said to be flat if its Riemann tensor vanishes everywhere. Otherwise it is said to be curved. The curvature tensor in Riemannian space is given by

$$R^{j}_{lhk} = \frac{\partial \Gamma^{j}{}_{lh}}{\partial x^{k}} - \frac{\partial \Gamma^{j}{}_{lk}}{\partial x^{h}} + \Gamma^{j}{}_{mk}\Gamma^{m}{}_{lh} - \Gamma^{j}{}_{mh}\Gamma^{m}{}_{lk} \quad \text{...3.31}$$

3.3.1 The properties of Riemann - Christoffel tensor

1. Anti -symmetric property. Interchanging the indices h and k in equation (3.31) we have

$$R^j_{lhk} = \frac{\partial \Gamma^j{}_{lk}}{\partial x^h} - \frac{\partial \Gamma^j{}_{lh}}{\partial x^k} + \Gamma^j{}_{mh}\Gamma^m{}_{lk} - \Gamma^j{}_{mk}\Gamma^m{}_{lh}$$

$$= -\left[\frac{\partial \Gamma^j{}_{lk}}{\partial x^h} - \frac{\partial \Gamma^j{}_{lh}}{\partial x^k} + \Gamma^j{}_{mk}\Gamma^m{}_{lh} - \Gamma^j_{mh}\Gamma^m_{lk} \right]$$

$$R^j_{lkh} = -R^j_{lhk}$$

Which shows that R^j_{lkh} is anti -symmetric with respect to the indices h and k

2. Cyclic property; permuting the indices l , h and k in a cyclic order and adding, we obtain the relation

$$R^j_{lhk} + R^j_{hkl} + R^j_{klh} = 0$$

3. Given a Riemann-Christoffel curvature tensor R^m_{jhk}, we can take the inner product with the metric tensor g_{lm} to obtain a type (0,4) R_{jlhk} that is

$$g_{lm} R^m_{jhk} = R_{jlhk}$$

This new type of tensor is usually called the covariant curvature tensor. It is derived as follows

From equation (3.31) replace j by m, l by j and m by p, we have

$$R^m_{jhk} = \frac{\partial \Gamma^m_{jh}}{\partial x^k} - \frac{\partial \Gamma^m_{jk}}{\partial x^h} + \Gamma^m_{pk}\Gamma^p_{jh} - \Gamma^m_{ph}\Gamma^p_{jk}$$

Taking the inner product of the above equation with g_{lm}

$$g_{lm} R^m_{jhk} = g_{lm} \frac{\partial \Gamma^m_{jh}}{\partial x^k} - g_{lm} \frac{\partial \Gamma^m_{jk}}{\partial x^h} + g_{lm}\Gamma^m_{pk}\Gamma^p_{jh} - g_{lm}\Gamma^m_{ph}\Gamma^p_{jk}$$

$$R_{jlhk} = \frac{\partial \Gamma_{jlh}}{\partial x^k} - \frac{\partial g_{lm}}{\partial x^k}\Gamma^m_{jh} - \frac{\partial \Gamma_{jlk}}{\partial x^h} + \frac{\partial g_{lm}}{\partial x^k}\Gamma^m_{jk} + \Gamma_{mlk}\Gamma^m_{jh} - \Gamma_{mlh}\Gamma^m_{jk}$$

But

$$\frac{\partial \Gamma_{jlh}}{\partial x^k} = \frac{1}{2}\frac{\partial}{\partial x^k}\left[\frac{\partial g_{lh}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^h} - \frac{\partial g_{jh}}{\partial x^l}\right]$$

$$= \frac{1}{2}\left[\frac{\partial^2 g_{lh}}{\partial x^k \partial x^j} + \frac{\partial^2 g_{lj}}{\partial x^k \partial x^h} - \frac{\partial^2 g_{jh}}{\partial x^k \partial x^l}\right]$$

also

$$\frac{\partial \Gamma_{jlk}}{\partial x^h} = \frac{1}{2}\frac{\partial}{\partial x^h}\left[\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right]$$

$$= \frac{1}{2}\left[\frac{\partial^2 g_{lh}}{\partial x^k \partial x^j} + \frac{\partial^2 g_{lj}}{\partial x^k \partial x^h} - \frac{\partial^2 g_{jh}}{\partial x^k \partial x^l}\right]$$

And

$$\frac{\partial \Gamma_{jlh}}{\partial x^k} - \frac{\partial \Gamma_{jlk}}{\partial x^h} = \frac{1}{2}\left[\frac{\partial^2 g_{lh}}{\partial x^k \partial x^j} - \frac{\partial^2 g_{lk}}{\partial x^h \partial x^j} + \frac{\partial^2 g_{jk}}{\partial x^h \partial x^l} - \frac{\partial^2 g_{jh}}{\partial x^k \partial x^l}\right]$$

$$\frac{\partial g_{lm}}{\partial x^k} = g_{pm}\Gamma^p_{lk} + g_{lp}\Gamma^p_{mk} = \Gamma_{lmk} + \Gamma_{mlk}$$

$$\frac{\partial g_{lm}}{\partial x^h} = g_{pm}\Gamma^p_{lh} + g_{lp}\Gamma^p_{mh} = \Gamma_{lmh} + \Gamma_{mlh}$$

$$R_{jlhk} = \frac{1}{2}\left[\frac{\partial^2 g_{lh}}{\partial x^k \partial x^j} - \frac{\partial^2 g_{lk}}{\partial x^h \partial x^j} + \frac{\partial^2 g_{jk}}{\partial x^h \partial x^l} - \frac{\partial^2 g_{jh}}{\partial x^k \partial x^l}\right] - \left(g_{pm}\Gamma^p_{lk} + g_{lp}\Gamma^p_{mk}\right)\Gamma^m_{jh}$$

$$+ \left(g_{pm}\Gamma^p_{lh} + g_{lp}\Gamma^p_{mh}\right)\Gamma^m_{jk} + \Gamma_{mlk}\Gamma^m_{jh} - \Gamma_{mlh}\Gamma^m_{jk}$$

$$R_{jlhk} = \frac{1}{2}\left[\frac{\partial^2 g_{lh}}{\partial x^k \partial x^j} - \frac{\partial^2 g_{lk}}{\partial x^h \partial x^j} + \frac{\partial^2 g_{jk}}{\partial x^h \partial x^l} - \frac{\partial^2 g_{jh}}{\partial x^k \partial x^l}\right] - g_{pm}\Gamma^p_{lk}\Gamma^m_{jh} - g_{lp}\Gamma^p_{mk}\Gamma^m_{jh} + g_{pm}\Gamma^p_{lh}\Gamma^m_{jk} +$$

$$g_{lp}\Gamma^p_{mh}\Gamma^m_{jk} + g_{lp}\Gamma^p_{mk}\Gamma^m_{jh} - g_{lp}\Gamma^p_{mh}\Gamma^m_{jk}$$

$$R_{jlhk} = \frac{1}{2}\left[\frac{\partial^2 g_{lh}}{\partial x^k \partial x^j} - \frac{\partial^2 g_{lk}}{\partial x^h \partial x^j} + \frac{\partial^2 g_{jk}}{\partial x^k \partial x^l} - \frac{\partial^2 g_{jh}}{\partial x^k \partial x^l}\right] - g_{pm}\Gamma^p_{lk}\Gamma^m_{jh} + g_{pm}\Gamma^p_{lh}\Gamma^m_{jk}$$

$$R_{jlhk} = \frac{1}{2}\left[\frac{\partial^2 g_{lh}}{\partial x^k \partial x^j} - \frac{\partial^2 g_{lk}}{\partial x^h \partial x^j} + \frac{\partial^2 g_{jk}}{\partial x^h \partial x^l} - \frac{\partial^2 g_{jh}}{\partial x^k \partial x^l}\right] + g_{pm}\left[\Gamma^m_{jk}\Gamma^p_{lh} - \Gamma^m_{jh}\Gamma^p_{lk}\right] \quad \text{......................3.32}$$

In general the curvature of a space is given by the Gaussian curvature K

Given by the formula $K = \frac{R_{1212}}{g}$ where g is the determinant of the spatial part of the metric and R_{1212} can be obtained from equation (3.32) as follows

Setting $j = h = 1$ and $l = k = 2$

$$R_{1212} = \frac{1}{2}\left[\frac{\partial^2 g_{21}}{\partial x^2 \partial x^1} - \frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} + \frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} - \frac{\partial^2 g_{11}}{\partial x^2 \partial x^2}\right] + g_{pm}\left[\Gamma^m_{12}\Gamma^p_{21} - \Gamma^m_{11}\Gamma^p_{22}\right]$$

Since $g_{12} = g_{21} = 0$ we have

$$R_{1212} = \frac{1}{2}\left[-\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} - \frac{\partial^2 g_{11}}{\partial x^2 \partial x^2}\right] + g_{pm}\left[\Gamma^m_{12}\Gamma^p_{21} - \Gamma^m_{11}\Gamma^p_{22}\right]$$

Summing over p and m, we obtain

$$R_{1212} = \frac{1}{2}\left[-\frac{\partial^2 g_{22}}{\left(\partial x^1\right)^2} - \frac{\partial^2 g_{11}}{\left(\partial x^2\right)^2}\right] + g_{11}\left[\Gamma^1_{12}\Gamma^1_{21} - \Gamma^1_{11}\Gamma^1_{22}\right] + g_{22}\left[\Gamma^2_{12}\Gamma^2_{21} - \Gamma^2_{11}\Gamma^2_{22}\right]$$

Substituting

$$\Gamma^1_{11} = \frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial x^1}, \quad \Gamma^2_{22} = \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial x^2}, \quad \Gamma^1_{12} = \frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial x^2}, \quad \Gamma^2_{21} = \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial x^1}, \quad \Gamma^1_{22} = -\frac{1}{2g_{11}}\frac{\partial g_{22}}{\partial x^1},$$

$$\Gamma^2_{11} = -\frac{1}{2g_{22}}\frac{\partial g_{11}}{\partial x^2}$$

$$R_{1212} = \frac{1}{2}\left[-\frac{\partial^2 g_{22}}{\left(\partial x^1\right)^2} - \frac{\partial^2 g_{11}}{\left(\partial x^2\right)^2}\right] + g_{11}\left[\left(\frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial x^2}\right)^2 + \frac{1}{4g^2_{11}}\frac{\partial g_{11}}{\partial x^1}\frac{\partial g_{22}}{\partial x^1}\right]$$

$$+g_{22}\left[\left(\frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial x^1}\right)^2 + \frac{1}{4g^2_{22}}\frac{\partial g_{11}}{\partial x^2}\frac{\partial g_{22}}{\partial x^2}\right]$$

$$R_{1212} = \frac{1}{2}\left[-\frac{\partial^2 g_{22}}{\left(\partial x^1\right)^2} - \frac{\partial^2 g_{11}}{\left(\partial x^2\right)^2}\right] + \frac{1}{4g_{11}}\left[\left(\frac{\partial g_{11}}{\partial x^2}\right)^2 + \frac{\partial g_{11}}{\partial x^1}\frac{\partial g_{22}}{\partial x^1}\right] + \frac{1}{4g_{22}}\left[\left(\frac{\partial g_{22}}{\partial x^1}\right)^2 + \frac{\partial g_{11}}{\partial x^2}\frac{\partial g_{22}}{\partial x^2}\right]$$

Hence using $K = \frac{R_{1212}}{g}$ we have

$$K = \frac{1}{2g}\left[-\frac{\partial^2 g_{22}}{\left(\partial x^1\right)^2} - \frac{\partial^2 g_{11}}{\left(\partial x^2\right)^2}\right] + \frac{g_{22}}{4g^2}\left[\left(\frac{\partial g_{11}}{\partial x^2}\right)^2 + \frac{\partial g_{11}}{\partial x^1}\frac{\partial g_{22}}{\partial x^1}\right] + \frac{g_{11}}{4g^2}\left[\left(\frac{\partial g_{22}}{\partial x^1}\right)^2 + \frac{\partial g_{11}}{\partial x^2}\frac{\partial g_{22}}{\partial x^2}\right]$$

(*Steven* Weinberg)[5]

With a given diagonal metric the curvature K can be calculated by using the formula

$$K = -\frac{1}{2g}\left[\frac{\partial^2 g_{22}}{\left(\partial x^1\right)^2} + \frac{\partial^2 g_{11}}{\left(\partial x^2\right)^2}\right] + \frac{g_{22}}{4g^2}\left[\left(\frac{\partial g_{11}}{\partial x^2}\right)^2 + \frac{\partial g_{11}}{\partial x^1}\frac{\partial g_{22}}{\partial x^1}\right] + \frac{g_{11}}{4g^2}\left[\left(\frac{\partial g_{22}}{\partial x^1}\right)^2 + \frac{\partial g_{11}}{\partial x^2}\frac{\partial g_{22}}{\partial x^2}\right]$$

3.4 The Robertson-walker metric in (3+1)-dimensions

The only metric that is in accord with the cosmological principle is the Robertson-Walker metric.

The Robertson-Walker metric in 4-dimensional space-time is given by

$$ds^2 = c^2dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right] \text{............................3.33}$$

Where R (t) is called the **expansion factor** or **cosmic scale factor** and k is the curvature scalar.

3.4.1 Derivation of the Robertson-walker metric in (3+1)-dimensions

The simplest way to derive this metric is to consider a 3 - dimensional space as a space embedded in a 4 - dimensional hypersurface.

Assuming that space-time is static, we can choose a time coordinate t such that the line element of space-time could be described by

$$ds^2 = c^2dt^2 - \alpha_{ij}dx^i dx^j \text{ ....3.34}$$

Where αij functions of space are coordinates $x^i (i, j = 1, 2, 3\, only)$

We can now construct a **homogeneous** and **isotropic** closed space of the positive curvature.

Let (x^1, x^2, x^3, x^4) be a rectangular Cartesian coordinates in E_4. Then a hypersphere of radius R has equation

$$(R)^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 \text{ ....3.35}$$

Where R is in general a function of time t

In such a space the line element is defined as

$$d\sigma^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 \text{.....................................3.36}$$

To use coordinates intrinsic to the surface, we set

$$x^1 = R\sin\chi\cos\theta$$

$$x^2 = R\sin\chi\sin\theta\cos\phi$$

$$x^3 = R\sin\chi\sin\theta\sin\phi$$

$$x^4 = R\cos\chi$$

Computing the $\left(dx^i\right)^2$, substituting into equation (3.36) and simplifying, we obtain

$$d\sigma^2 = R^2 d\chi^2 + R^2\sin^2\chi d\theta^2 + R^2\sin^2\chi\sin^2\theta d\phi^2$$

Hence

$$d\sigma^2 = R^2(d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi\sin^2\theta d\phi^2) \text{ ....3.37}$$

The ranges of θ, ϕ and χ are given by

$$0 \le \chi \le \pi \ , \qquad 0 \le \theta \le \pi \ , \qquad 0 \le \phi \le 2\pi$$

Setting $r = \sin\chi$ and differentiating we have

$$dr^2 = \cos^2\chi d\chi^2$$

$$d\chi^2 = \frac{dr^2}{\cos^2\chi} = \frac{dr^2}{1-r^2} \text{ ....3.38}$$

Substituting (3.38) into (3.37), we have

$$d\sigma^2 = R^2\left[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \text{ ....3.39}$$

Where R is the radius of the universe or the expansion factor

Substituting (3.39) into (3.34), the line element becomes

$$ds^2 = c^2dt^2 - R^2\left[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \text{ ....3.40}$$

This is the Robertson –Walker metric for a space of constant positive curvature.

We can similarly get other homogeneous and isotropic spaces by considering them as 3 – surfaces of constant negative curvature. In terms of the Cartesian coordinates x^1, x^2, x^3, x^4 used earlier, a 3 – surface of constant negative curvature is given by an equation of the form

$$(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 = -(R)^2$$

Where R is in general a function of time

The line element in this pseudo-Euclidean space is given by

$$d\sigma^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2 \quad \text{...3.41}$$

By setting

$$x^1 = R\sinh\chi\cos\theta,$$

$$x^2 = R\sinh\chi\sin\theta\cos\phi,$$

$$x^3 = R\sinh\chi\sin\theta\sin\phi,$$

$$x^4 = R\cosh\chi$$

Equation (3.41) becomes

$$d\sigma^2 = R^2\left[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right] \quad \text{.............................3.42}$$

The negative sign on $(dx^4)^2$ means we are embedding our 3 – surface in a pseudo – Euclidean space but not in a Euclidean space.

In Euclidean space, the Pythagoras theorem holds with the line – element given by $dx^2 = dx_1{}^2 + dx_2{}^2 + dx_3{}^2 + ...$ If some of the plus sign on the right – hand side are changed to minus signs, the results is a pseudo – Euclidean space. Thus **Minkowski** space is a pseudo-Euclidean space.

By setting $r = \sinh\chi$

$$dr = \cosh\chi d\chi$$

$$\frac{dr^2}{\cosh^2 \chi} = d\chi^2$$

$$\frac{dr^2}{1+r^2} = d\chi^2 \quad \text{...3.43}$$

Substituting (3.43) into (3.42), we obtain

$$d\sigma^2 = R^2\left[\frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \quad \text{..3.44}$$

Comparing (3.44) with (3.39), we can combine both expressions into a single expression by introducing a parameter k that takes values ±1. That is

$$d\sigma^2 = R^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \quad \text{...3.45}$$

Putting k = 0 in (3.45), we obtain

$$d\sigma^2 = R^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

Which is the Euclidean line element scaled by the constant factor R. This is the third alternative of the 3 – surface of zero curvature. The most general line element which is referred to as the Robertson – Walker line element is given by

$$ds^2 = c^2dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

Where R (t) is called the **expansion factor** or **cosmic scale factor** and k is the curvature **parameter**

The main reason for using Robertson – Walker metric for this project is that, it was formulated based on the cosmological principle. That is a space is ***homogeneous*** and ***isotropic.***

Isotropic condition tells us that there should be no terms of the form $dtdx^{\mu}$ in the line element. This can easily be seen in the following way. If we had terms like $g_{o\mu}dtdx^{\mu}$ in the line element, then spatial displacements dx^{μ} and $-dx^{\mu}$ would contribute oppositely to ds^2 over a small time interval dt, and such directional variation is forbidden by isotropy.

The (**figure 3.3**) below shows three surfaces formed when the curvature scalar is varied from -1, 0 to +1.

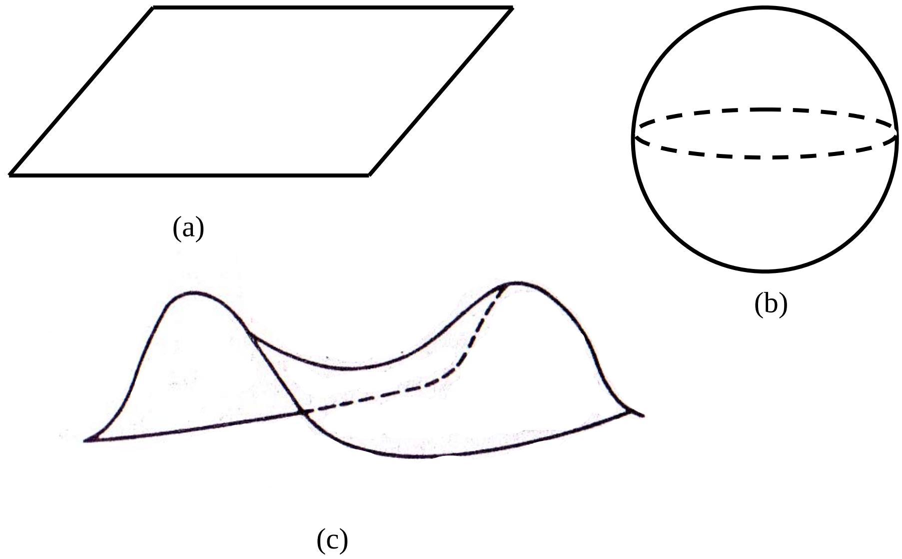

Figure 3.3 (a) shows a section of the Euclidean plane, (b) shows a spherical surface,

(c) Shows a saddle – shaped surface

Suppose we try to cover these surfaces with a plain sheet of paper. We will find that our sheet fits exactly and smoothly on the plane surface. If we try to cover the spherical surface, the sheet of paper develops wrinkles, indicating that the sheet of paper has area in excess of that needed to cover the surface. Similarly, in trying to cover the saddle our paper will be torn, being short of the necessary covering area. These differences can be expressed in differential geometry by the notion of curvature. The plane surface has zero curvature, the spherical surface has positive curvature and the saddle has negative curvature. Our paper covering experiment tells us in general whether a given surface has a zero, positive or negative curvature as stated by (*Jayant* V Narliker)[1]

Chapter 4

4.1 The derivation of the Robertson-Walker metric in (2+1) dimensions

In analogy with the Robertson walker metric, we consider a 2-dimensional space as a space embedded in 3-dimensional hypersurface

Assuming that the space time is static so that we can choose a time coordinate t such that the line element of space time could be described by

$$ds^2 = c^2 dt^2 - a_{ij} dx^i dx^j \quad \text{...4.0}$$

Where a_{ij} functions of space are coordinates $x^i (i, j = 1, 2)$ .We can construct a homogenous and isotropic closed space of positive curvature.

Let (x^1, x^2, x^3) be a rectangular Cartesian coordinates in E_3. Then a hypersphere of radius R has an equation

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = (R)^2 \quad \text{...4.1}$$

where R is in general a function of time

Let the line element in such a space be given by

$$d\sigma^2 = a_{ij} dx^i dx^j \text{ or}$$

$$d\sigma^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \quad \text{..4.2}$$

Then by setting

$$x^1 = R \sin\chi \cos\theta$$

$$x^2 = R\sin\chi\sin\theta$$

$$x^3 = R\cos\chi$$

We Compute the $\left(dx^i\right)^2$, substitute into (**4.2**) and simplify, we obtain

$$d\sigma^2 = R^2 d\chi^2 + R^2 \sin^2 \chi d\theta^2$$

Setting $r = \sin\chi \quad dr = \cos\chi d\chi$

$$d\chi^2 = \frac{dr^2}{\cos^2\chi} = \frac{dr^2}{1-\sin^2\chi} = \frac{dr^2}{1-r^2}$$

Hence

$$d\sigma^2 = R^2\left[\frac{dr^2}{1-r^2} + r^2 d\theta^2\right] \quad \text{..4.3}$$

Substituting equation (4.3) into (4.0), we obtain

$$ds^2 = c^2dt^2 - R^2\left[\frac{dr^2}{1-r^2} + r^2 d\theta^2\right] \quad \text{.. 4.4}$$

This is the Robertson-Walker metric for a space of constant positive curvature

Similarly, we can construct a homogeneous and isotropic space by considering them as 2-surfaces of constant negative curvature.

In terms of the Cartesian coordinates x^1, x^2, x^3 used earlier, two surfaces of constant negative curvature is given by

$$(x^1)^2 + (x^2)^2 - (x^3)^2 = -(R)^2 \quad \text{...............................4.5}$$

Where R is in general a function of time

The line element in this pseudo-Euclidean space is given by

$$d\sigma^2 = (dx^1)^2 + (dx^2)^2 - (dx^3)^2 \quad \text{... 4.6}$$

Now setting

$$x^1 = R\sin h\chi\cos\theta$$

$$x^2 = R\sinh\chi\sin\theta$$

$$x^3 = R\cosh\chi$$

Computing the $\left(dx^i\right)^2$, and substituting into (**4.6**) and simplifying, we obtain

$$d\sigma^2 = R^2 d\chi^2 + R^2\sinh^2\chi d\theta^2$$

$$d\sigma^2 = R^2(d\chi^2 + \sinh^2\chi d\theta^2) \quad \text{...} \; 4.7$$

Setting $r = \sinh\chi$

$$dr = \cosh\chi d\chi$$

$$dr^2 = \cosh^2\chi d\chi^2$$

$$dx^2 = \frac{dr^2}{\cosh^2\chi} = \frac{dr^2}{1+\sinh^2\chi} = \frac{dr^2}{1+r^2}$$

$$d\sigma^2 = R^2\left[\frac{dr^2}{1+r^2} + r^2 d\theta^2\right] \quad \text{..} 4.8$$

Combining equations (**4.7**) and (**4.8**) introducing a parameter k that takes the values ± 1 and 0, we have

$$d\sigma^2 = R^2\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2\right] \quad \text{...} 4.9$$

When $k = 0$

$$d\sigma^2 = R^2\left[dr^2 + r^2 d\theta^2\right]$$

Which is the Euclidean line element scaled by a constant factor. It is the third surface of zero curvature.

The Robertson Walker like line element in (**2+1**)-dimension is given by

$$ds^2 = c^2dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2\right] \text{.....................................4.10}$$

Where R (t) is the expansion factor and k is the curvature parameter that characterises the geodesic of the space.

4.2 Calculation of the curvature of (2+1)-dimensional space

From equation (**4.10**) the determinant of the special part is given by

$$g_{ij} = \begin{pmatrix} \frac{R^2}{1-kr^2} & 0 \\ 0 & R^2r^2 \end{pmatrix}$$

Using the formula derived from section **3.3.1,** the curvature K is given by

$$K = \frac{1}{2g}\left[-\frac{\partial^2 g_{22}}{\left(\partial x^1\right)^2} - \frac{\partial^2 g_{11}}{\left(\partial x^2\right)^2}\right] + \frac{g_{22}}{4g^2}\left[\left(\frac{\partial g_{11}}{\partial x^2}\right)^2 + \frac{\partial g_{11}}{\partial x^1}\frac{\partial g_{22}}{\partial x^1}\right] + \frac{g_{11}}{4g^2}\left[\left(\frac{\partial g_{22}}{\partial x^1}\right)^2 + \frac{\partial g_{11}}{\partial x^2}\frac{\partial g_{22}}{\partial x^2}\right]$$

Where $g_{11} = \frac{R^2}{1-kr^2}$ $g_{22} = R^2r^2$ and $g = \frac{R^4r^2}{1-kr^2}$ is the determinant of the metric tensor taken $x^1 = r$, $x^2 = \theta$

$$\frac{\partial g_{22}}{\partial x^1} = \frac{\partial g_{22}}{\partial r} = 2R^2r$$

$$\frac{\partial^2 g_{22}}{\left(\partial x^1\right)^2} = \frac{\partial^2 g_{22}}{\partial r^2} = 2R^2$$

$$\frac{\partial g_{11}}{\partial x^2} = \frac{\partial g_{22}}{\partial x^2} = 0$$

$$\frac{\partial g_{11}}{\partial x^1} = \frac{\partial g_{11}}{\partial r} = \frac{2kR^2r}{\left(1-kr^2\right)^2}$$

Hence

$$K=\frac{1}{2g}\left[-\frac{\partial^2 g_{22}}{\left(\partial x^1\right)^2}\right]+\frac{g_{22}}{4g^2}\left[\frac{\partial g_{11}}{\partial x^1}\frac{\partial g_{22}}{\partial x^1}\right]+\frac{g_{11}}{4g^2}\left[\left(\frac{\partial g_{22}}{\partial x^1}\right)^2\right]$$

$$K=\frac{\left(1-kr^2\right)}{2R^4r^2}\left(-2R^2\right)+\frac{R^2r^2}{4}\frac{\left(1-kr^2\right)^2}{R^8r^4}\left(\frac{2kR^2r}{\left(1-kr^2\right)^2}2R^2r\right)+\frac{R^2}{4\left(1-kr^2\right)}\frac{\left(1-kr^2\right)^2}{R^8r^4}\left(4R^4r^2\right)$$

$$K=\frac{-\left(1-kr^2\right)}{R^2r^2}+\frac{k}{R^2}+\frac{\left(1-kr^2\right)}{R^2r^2}=\frac{k}{R^2}$$

Where $k=0$ defines a flat space, $k=-1$ defines a space of negative curvature and $k=+1$ a space of positive curvature

4.3 Calculations of the christoffel symbols

Consider the line element

$$ds^2 = c^2dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2\right]$$

The metric determinant from equation (4.10**)** is given by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{-R^2(t)}{1-kr^2} & 0 \\ 0 & 0 & -r^2R^2(t) \end{pmatrix}$$

Let $x^1 = ct$, $x^2 = r$ and $x^3 = \theta$

In our calculations we shall write R (t) as R and $\dot{R} = \dfrac{dR}{dt}$ for short, R is a function of t only.

Using the formulae derived from section 2.4

The christoffel symbols are given by

(1) $$\gamma_{ii}^{i} = \frac{1}{2g_{ii}}\frac{\partial g_{ii}}{\partial x^i}$$

$$\gamma_{ii}^{i} = \frac{1}{2g_{ii}}\frac{\partial g_{ii}}{\partial x^i} = 0 \quad \text{for } i = 1, 3$$

$$\gamma_{22}^{2} = \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial x^2} = \frac{1-kr^2}{-2R^2}\frac{\partial\left(\dfrac{-R^2}{1-kr^2}\right)}{\partial r} = \frac{1-kr^2}{-2R^2}\left(\frac{-2krR^2}{\left(1-kr^2\right)^2}\right) = \frac{kr}{1-kr^2}$$

(2) $$\gamma_{ij}^{i} = \frac{1}{2g_{ii}}\frac{\partial g_{ii}}{\partial x^j}$$

$$\gamma_{1j}^{1} = 0 \text{ for } j = 2, 3$$

$$\gamma_{21}^{2} = \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial x^1}$$

$$= \frac{1}{2\left(\frac{-R^2}{1-kr^2}\right)} \frac{\partial\left(\frac{-R^2}{1-kr^2}\right)}{c\partial t} = \frac{1-kr^2}{-2R^2}\left(\frac{-2R\dot{R}}{1-kr^2}\right) = \frac{\dot{R}}{cR}$$

$$\gamma_{23}^2 = \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial x^3} = 0$$

$$\gamma_{31}^3 = \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^1} = \frac{1}{-2R^2r^2} \frac{\partial\left(-R^2r^2\right)}{c\partial t} = \frac{1}{-2R^2r^2}\left(\frac{-2Rr^2\dot{R}}{c}\right) = \frac{\dot{R}}{cR}$$

$$\gamma_{32}^3 = \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^2} = \frac{1}{-2R^2r^2} \frac{\partial\left(-R^2r^2\right)}{\partial r} = \frac{-2R^2r}{-2R^2r^2} = \frac{1}{r}$$

(3) $$\gamma_{jj}^i = \frac{-1}{2g_{ii}} \frac{\partial g_{jj}}{\partial x^i}$$

$$\gamma_{22}^1 = \frac{-1}{2g_{11}} \frac{\partial g_{22}}{\partial x^1} = -\frac{1}{2} \frac{\partial\left(\frac{-R^2}{1-kr^2}\right)}{c\partial t} = \frac{1}{2c}\left(\frac{2R\dot{R}}{1-kr^2}\right) = \frac{R\dot{R}}{c(1-kr^2)}$$

$$\gamma_{33}^1 = \frac{-1}{2g_{11}} \frac{\partial g_{33}}{\partial x^1} = \frac{1}{2} \frac{\partial\left(-R^2r^2\right)}{c\partial t} = \frac{1}{2}\left(\frac{2R\dot{R}r^2}{c}\right) = \frac{R\dot{R}r^2}{c}$$

$$\gamma_{11}^2 = \frac{-1}{2g_{22}} \frac{\partial g_{11}}{\partial x^2} = 0$$

$$\gamma_{33}^2 = \frac{-1}{2g_{22}} \frac{\partial g_{33}}{\partial x^2} = -\frac{1}{2\left(\frac{-R^2}{1-kr^2}\right)} \frac{\partial\left(-R^2r^2\right)}{\partial r} = \frac{1-kr^2}{2R^2}\left(-2R^2r\right) = -r\left(1-kr^2\right)$$

$\gamma_{jj}^3 = 0$ for $j = 1, 2$

The non-vanishing christoffel symbols therefore are

$$\gamma_{22}^1 = \frac{R\dot{R}}{C(1-kr^2)} \quad , \gamma_{33}^1 = \frac{R\dot{R}r^2}{c}, \quad \gamma_{22}^2 = \frac{kr}{1-kr^2} \;, \gamma_{21}^2 == \frac{\dot{R}}{cR}, \quad \gamma_{33}^2 = -r\left(1-kr^2\right),$$

$$\gamma_{31}^3 = \frac{\dot{R}}{cR} \;, \quad \gamma_{32}^3 = \frac{1}{r}$$

4.4 Determination of the geodesics

Using the general geodesic equation from section **3.2**

$$\frac{d^2x^j}{ds^2} + \gamma^j_{hk}\frac{dx^h}{ds}\frac{dx^k}{ds} = 0 \text{ ....4.11}$$

The differential equations are formulated as follows

For $j = 1$ we have (**4.11**) to be

$$\frac{d^2x^1}{ds^2} + \gamma^1_{22}\left(\frac{dx^2}{ds}\right)^2 + \gamma^1_{33}\left(\frac{dx^3}{ds}\right)^2 = 0$$

$$c\frac{d^2t}{ds^2} + \frac{R\dot{R}}{c(1-kr^2)}\left(\frac{dr}{ds}\right)^2 + \frac{R\dot{R}r^2}{c}\left(\frac{d\theta}{ds}\right)^2 = 0 \text{ ....4.12}$$

For $j = 2$ equation (**4.11**) becomes

$$\frac{d^2x^2}{ds^2} + \gamma^2_{21}\frac{dx^2}{ds}\frac{dx^1}{ds} + \gamma^2_{22}\left(\frac{dx^2}{ds}\right)^2 + \gamma^2_{33}\left(\frac{dx^3}{ds}\right)^2 = 0$$

$$\frac{d^2r}{ds^2} + \frac{2\dot{R}}{cR}\frac{dr}{ds}\frac{cdt}{ds} + \frac{kr}{1-kr^2}\left(\frac{dr}{ds}\right)^2 - r(1-kr^2)\left(\frac{d\theta}{ds}\right)^2 = 0$$

Hence

$$\frac{d^2r}{ds^2} + \frac{2\dot{R}}{R}\frac{dr}{ds}\frac{dt}{ds} + \frac{kr}{1-kr^2}\left(\frac{dr}{ds}\right)^2 - r(1-kr^2)\left(\frac{d\theta}{ds}\right)^2 = 0 \text{ ....4.13}$$

For $j = 3$ equation (**4.11**) becomes

$$\frac{d^2x^3}{ds^2} + \gamma^3_{31}\frac{dx^3}{ds}\frac{dx^1}{ds} + \gamma^3_{32}\frac{dx^3}{ds}\frac{dx^2}{ds} = 0$$

$$\frac{d^2\theta}{ds^2} + \frac{2\dot{R}}{cR}\frac{d\theta}{ds}\frac{cdt}{ds} + \frac{2}{r}\frac{d\theta}{ds}\frac{dr}{ds} = 0$$

Hence

$$\frac{d^2\theta}{ds^2}+\frac{2\dot{R}}{R}\frac{d\theta}{ds}\frac{dt}{ds}+\frac{2}{r}\frac{d\theta}{ds}\frac{dr}{ds}=0 \quad \text{...4.14}$$

The three differential equations to be solved are

$$c\frac{d^2t}{ds^2}+\frac{R\dot{R}}{c(1-kr^2)}\left(\frac{dr}{ds}\right)^2+\frac{R\dot{R}r^2}{c}\left(\frac{d\theta}{ds}\right)^2=0$$

$$\frac{d^2r}{ds^2}+\frac{2\dot{R}}{R}\frac{dr}{ds}\frac{dt}{ds}+\frac{kr}{1-kr^2}\left(\frac{dr}{ds}\right)^2-r(1-kr^2)\left(\frac{d\theta}{ds}\right)^2=0$$

$$\frac{d^2\theta}{ds^2}+\frac{2\dot{R}}{R}\frac{d\theta}{ds}\frac{dt}{ds}+\frac{2}{r}\frac{d\theta}{ds}\frac{dr}{ds}=0$$

From equation (**4.14)** the left hand side can be combined as follows

$$\frac{d}{ds}\left(R^2r^2\frac{d\theta}{ds}\right)=R^2r^2\frac{d^2\theta}{ds^2}+2R\dot{R}r^2\frac{d\theta}{ds}\frac{dt}{ds}+2rR^2\frac{d\theta}{ds}\frac{dr}{ds}$$

$$=R^2r^2\left(\frac{d^2\theta}{ds^2}+\frac{2\dot{R}}{R}\frac{d\theta}{ds}\frac{dt}{ds}+\frac{2}{r}\frac{d\theta}{ds}\frac{dr}{ds}\right)=0$$

Hence

$$\frac{d}{ds}\left(R^2r^2\frac{d\theta}{ds}\right)=0$$

$$\frac{dr}{ds}\frac{d}{dr}\left(R^2r^2\frac{d\theta}{ds}\right)=0$$

But $\qquad \frac{dr}{ds}\neq 0 \quad \frac{d}{dr}\left(R^2r^2\frac{d\theta}{ds}\right)=0$

$$\int\frac{d}{dr}\left(R^2r^2\frac{d\theta}{ds}\right)dr=0$$

$R^2r^2\frac{d\theta}{ds}=a$ where a is a constant not a fucntion of r

$$\frac{d\theta}{ds} = \frac{a}{R^2 r^2} \quad ....4.15$$

From equation **(4.13)** we have

$$\frac{d^2 r}{ds^2} + \frac{2\dot{R}}{R}\frac{dr}{ds}\frac{dt}{ds} + \frac{kr}{1-kr^2}\left(\frac{dr}{ds}\right)^2 = r(1-kr^2)\left(\frac{d\theta}{ds}\right)^2 \quad ....4.16$$

Combining the left hand side of the above equation we have

$$\frac{d}{ds}\left(\frac{R^2}{(1-kr^2)^{\frac{1}{2}}}\frac{dr}{ds}\right) = \frac{R^2}{(1-kr^2)^{\frac{1}{2}}}\frac{d^2 r}{ds^2} + \frac{2R\dot{R}}{(1-kr^2)^{\frac{1}{2}}}\frac{dr}{ds}\frac{dt}{ds} + \frac{R^2 kr}{(1-kr^2)^{\frac{3}{2}}}\left(\frac{dr}{ds}\right)^2$$

That is

$$\frac{d}{ds}\left(\frac{R^2}{(1-kr^2)^{\frac{1}{2}}}\frac{dr}{ds}\right) = \frac{R^2}{(1-kr^2)^{\frac{1}{2}}}\left(\frac{d^2 r}{ds^2} + \frac{2\dot{R}}{R}\frac{dr}{ds}\frac{dt}{ds} + \frac{kr}{1-kr^2}\left(\frac{dr}{ds}\right)^2\right)$$

$$\frac{\left(1-kr^2\right)^{\frac{1}{2}}}{R^2}\frac{d}{ds}\left(\frac{R^2}{(1-kr^2)^{\frac{1}{2}}}\frac{dr}{ds}\right) = \frac{d^2 r}{ds^2} + \frac{2\dot{R}}{R}\frac{dr}{ds}\frac{dt}{ds} + \frac{kr}{1-kr^2}\left(\frac{dr}{ds}\right)^2 \quad ....4.17$$

Hence equating (**4.16**) and (**4.17**) we have

$$\frac{\left(1-kr^2\right)^{\frac{1}{2}}}{R^2}\frac{d}{ds}\left(\frac{R^2}{(1-kr^2)^{\frac{1}{2}}}\frac{dr}{ds}\right) = r\left(1-kr^2\right)\left(\frac{d\theta}{ds}\right)^2$$

Multiplying through by $\frac{R^4}{1-kr^2}$, we have

$$\frac{R^2}{\left(1-kr^2\right)^{\frac{1}{2}}}\frac{dr}{ds}\frac{d}{dr}\left(\frac{R^2}{(1-kr^2)^{\frac{1}{2}}}\frac{dr}{ds}\right) = R^4 r\left(\frac{d\theta}{ds}\right)^2$$

Substituting equation (**4.15**) we have

$$\frac{R^2}{\left(1-kr^2\right)^{\frac{1}{2}}}\frac{dr}{ds}\frac{d}{dr}\left(\frac{R^2}{(1-kr^2)^{\frac{1}{2}}}\frac{dr}{ds}\right)=R^4r\left(\frac{a}{R^2r^2}\right)^2=\frac{a^2}{r^3} \quad \text{..........................4.18}$$

Let

$$p=\frac{R^2}{\left(1-kr^2\right)^{\frac{1}{2}}}\frac{dr}{ds} \quad \text{...4.19}$$

Equation (**4.18** **)** becomes

$$p\frac{dp}{dr}=\frac{a^2}{r^3}$$

$$\frac{dp^2}{2dr}=\frac{a^2}{r^3}$$

$$p^2=-\frac{a^2}{r^2}+\beta \quad \text{Where } \beta \text{ is a constant}$$

$$p^2=\frac{\beta r^2-a^2}{r^2}$$

$$p=\frac{1}{r}\sqrt{\beta r^2-a^2} \quad \text{.. 4.20}$$

Substituting equation (**4.20)** into (**4.19**) we have

$$\frac{R^2}{\left(1-kr^2\right)^{\frac{1}{2}}}\frac{dr}{ds}=\frac{1}{r}\sqrt{\beta r^2-a^2}$$

And

$$\frac{dr}{ds}=\frac{\left(1-kr^2\right)^{\frac{1}{2}}\left(\beta r^2-a^2\right)}{rR^2} \quad \text{..4.21}$$

4.4.1 The differential equation involving the coordinates r and θ

From equation (**4.21**) we have

$$\frac{d\theta}{ds}\frac{dr}{d\theta} = \frac{\left(1-kr^2\right)^{\frac{1}{2}}\left(\beta r^2 - a^2\right)}{rR^2}$$

$$\frac{dr}{d\theta} = \frac{\left(1-kr^2\right)^{\frac{1}{2}}\left(\beta r^2 - a^2\right)}{rR^2}\frac{ds}{d\theta} \quad \text{...4.22}$$

Substituting equation (**4.15**) we have

$$\frac{dr}{d\theta} = \frac{\left(1-kr^2\right)^{\frac{1}{2}}\left(\beta r^2 - a^2\right)}{rR^2}\frac{r^2R^2}{a}$$

$$\frac{dr}{d\theta} = \frac{r}{a}\left(1-kr^2\right)^{\frac{1}{2}}\left(\beta r^2 - a^2\right)^{\frac{1}{2}}$$

$$\frac{dr}{d\theta} = r\left(\frac{\beta}{a^2}r^2 - 1\right)^{\frac{1}{2}}\left(1-kr^2\right)^{\frac{1}{2}}$$

$$\frac{dr}{d\theta} = r\left(\alpha r^2 - 1\right)^{\frac{1}{2}}\left(1-kr^2\right)^{\frac{1}{2}} \quad \text{..4.23}$$

Where $\alpha = \frac{\beta}{a^2}$

$$\frac{dr}{d\theta} = r\sqrt{\left(\alpha r^2 - 1\right)\left(1-kr^2\right)}$$

$$d\theta = \frac{dr}{r\sqrt{\left(\alpha r^2 - 1\right)\left(1-kr^2\right)}}$$

$$d\theta = \frac{dr}{r^3\sqrt{\left(\alpha - \frac{1}{r^2}\right)\left(\frac{1}{r^2} - k\right)}}$$

$$d\theta = \frac{-\frac{1}{2} d\left(\frac{1}{r^2}\right)}{\sqrt{\left(\alpha - \frac{1}{r^2}\right)\left(\frac{1}{r^2} - k\right)}}$$

Set $u = \frac{1}{r^2}$ we have

$$\int d\theta = \int \frac{-\frac{1}{2} du}{\sqrt{(\alpha - u)(u - k)}}$$

Now set

$$u = \alpha \cos^2 \phi + k \sin^2 \phi \quad \text{...4.24}$$

Then

$$du = (-2\alpha \cos\phi \sin\phi + 2k \cos\phi \sin\phi)\, d\phi$$

$$du = -2\cos\phi \sin\phi (\alpha - k)\, d\phi$$

And

$$\sqrt{(\alpha - u)(u - k)} = \sqrt{\left(\alpha - \alpha\cos^2\phi - k\sin^2\phi\right)\left(\alpha\cos^2\phi + k\sin^2\phi - k\right)}$$

$$= \sqrt{\left(\alpha - \alpha\left(1 - \sin^2\phi\right) - k\sin^2\phi\right)\left(\alpha\cos^2\phi + k\left(1 - \cos^2\phi - k\right)\right)}$$

$$= \sqrt{\left(\alpha - \alpha + (\alpha - k)\sin^2\phi\right)\left((\alpha - k)\cos^2\phi + k - k\right)}$$

$$= (\alpha - k)\sin\phi\cos\phi$$

Hence

$$d\theta = \frac{-\frac{1}{2}(\alpha - k)(-2\cos\phi\sin\phi)\, d\phi}{(\alpha - k)\cos\phi\sin\phi} = d\phi$$

$$\int d\theta = \int d\phi$$

$$\theta = \phi \quad \text{...4.25}$$

From (**4.24**)

$$u = \alpha \cos^2 \theta + k \sin^2 \theta$$

$$= \alpha \cos^2 \theta + k\left(1 - \cos^2 \theta\right)$$

$$= \alpha \cos^2 \theta + k - k \cos^2 \theta$$

$$u - k = \left(\alpha - k\right)\cos^2 \theta$$

$$\frac{u-k}{\alpha - k} = \cos^2 \theta$$

$$\frac{\frac{1}{r^2} - k}{\alpha - k} = \cos^2 \theta$$

$$\frac{1}{r^2} = \left(\alpha - k\right)\cos^2 \theta + k$$

$$1 = \left(\alpha - k\right) r^2 \cos^2 \theta + kr^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; 4.26$$

4.4.2 The geodesics obtained in (2+1)-dimensions.

When $k=0$**, equation (4.26) becomes**

$\alpha r^2 \cos^2\theta = 1 \quad \alpha \geq 1$ and $0 \leq r \geq 1$

From the substitution

$$x_1 = R\sin\chi\cos\theta \qquad r = \sin\chi$$
$$x_1 = Rr\cos\theta$$
$$x_1 = \frac{R}{\sqrt{\alpha}}$$

R is a function of time but can be taken to be a constant in any particular epoch.

This is a straight line in Euclidean plane which describes an open two dimensional space

An epoch is a time frame of about one million years which is very small compare to the estimated age of the universe.

When $k=1$**, equation (4. 26) becomes**

$$1 = (\alpha - 1) r^2 \cos^2\theta + r^2 \quad \textbf{where} \quad \alpha \geq 1 \text{ and } 0 \leq \mathrm{r} \leq 1$$

From the substitution

$$x_1 = R\sin\chi\cos\theta \quad x_2 = R\sin\chi\sin\theta \text{ but } r = \sin\chi$$

$$x_1 = Rr\cos\theta \quad x_2 = Rr\sin\theta \qquad \frac{x_1 + x_2}{R^2} = r^2$$

$$1 = (\alpha - 1)\frac{x_1}{R^2} + \frac{x_1 + x_2}{R^2}$$

$R^2 = \alpha x_1^2 + x_2^2$ **or**

$$\alpha x^2 + y^2 = R^2$$

Graphs of different values of α

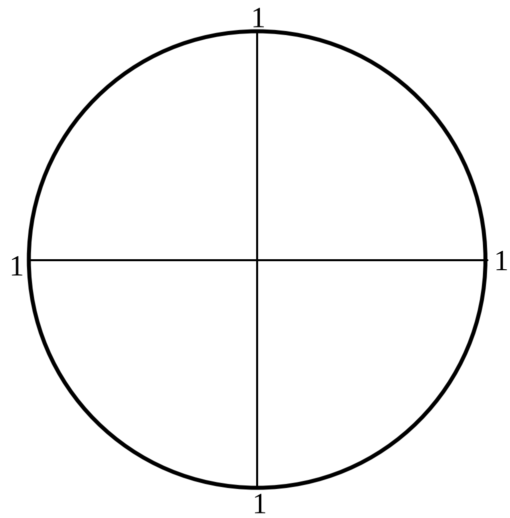

Figure 4.1 Graph of $\alpha = 1$ and $R = 1$ we have $x^2 + y^2 = 1$ indicating a great circle

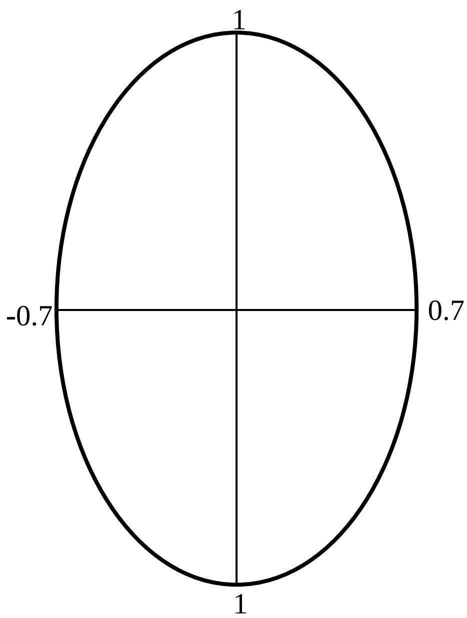

Figure 4.2 Graph of $\alpha > 1$, indicating ellipses.

When $k = -1$, equation (4.26) becomes

$1 = (\alpha + 1) r^2 \cos^2 \theta - r^2$ where $\alpha \geq 1$ and $0 \leq r \leq 1$

Using the transformations

$$x_1 = x = R\sin\chi\cos\theta \quad x_2 = y = R\sin\chi\sin\theta \ \textbf{and} \ r = \sin\chi$$

$$1 = (\alpha+1)\frac{x^2}{R^2} - \frac{x^2+y^2}{R^2}$$

$$R^2 = \alpha x^2 - y^2$$

Graphs of different values of α

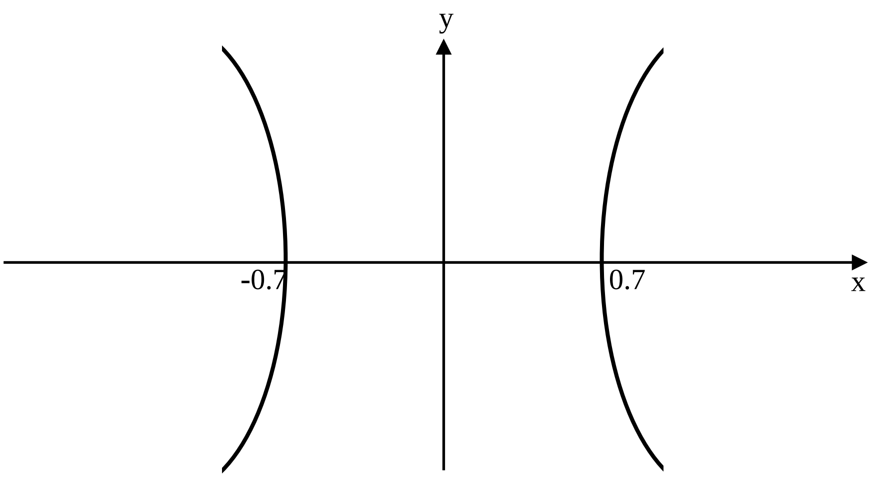

Figure 4.3 Plotting with different values of α show different hyperbolae, they describe the surface of a saddle and are open spaces.

4.5 The Null Geodesic

The differential equations in terms of the affine parameter are

$$c^2\frac{d^2t}{d\lambda^2}+\frac{R\dot{R}}{1-kr^2}\left(\frac{dr}{d\lambda}\right)^2+R\dot{R}r^2\left(\frac{d\theta}{d\lambda}\right)^2=0 \quad \text{..4.27}$$

$$\frac{d^2r}{d\lambda^2}+2\frac{\dot{R}}{R}\frac{dr}{d\lambda}\frac{dt}{d\lambda}+\frac{kr}{1-kr^2}\left(\frac{dr}{d\lambda}\right)^2-r\left(1-kr^2\right)\left(\frac{d\theta}{d\lambda}\right)^2=0 \quad \text{..........................4.28}$$

$$\frac{d^2\theta}{d\lambda^2}+2\frac{\dot{R}}{R}\frac{d\theta}{d\lambda}\frac{dt}{d\lambda}+\frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda}=0 \quad \text{...4.29}$$

From **(4.29)** we have

$$\frac{d}{d\lambda}\left(R^2r^2\frac{d\theta}{d\lambda}\right)=R^2r^2\frac{d^2\theta}{d\lambda^2}+2R\dot{R}r^2\frac{dt}{d\lambda}\frac{d\theta}{d\lambda}+2rR^2\frac{d\theta}{d\lambda}\frac{dr}{d\lambda}$$

$$=R^2r^2\left[\frac{d^2\theta}{d\lambda^2}+2\frac{\dot{R}}{R}\frac{dt}{d\lambda}\frac{d\theta}{d\lambda}+\frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda}\right]=0$$

$$\frac{d}{d\lambda}\left(R^2r^2\frac{d\theta}{d\lambda}\right)=0$$

$$\frac{dr}{d\lambda}\frac{d}{dr}\left(R^2r^2\frac{d\theta}{d\lambda}\right)=0$$

Since $\frac{dr}{d\lambda}\neq 0\,,\qquad \frac{d}{dr}\left(R^2r^2\frac{d\theta}{d\lambda}\right)=0$

And therefore

$$R^2 r^2 \frac{d\theta}{d\lambda} = a$$ where a is a constant independent of r

Hence $$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = \frac{a}{R^2 r^2}$$ 4.30

For a null geodesic, we have

$$g_{ii}\left(\frac{dx^i}{d\lambda}\right)^2 = 0$$

Thus,

$$g_{11}\left(\frac{dx^1}{d\lambda}\right)^2 + g_{22}\left(\frac{dx^2}{d\lambda}\right)^2 + g_{33}\left(\frac{dx^3}{d\lambda}\right)^2 = 0$$

Substituting the metric, we have

$$c^2\left(\frac{dt}{d\lambda}\right)^2 - \frac{R^2}{1-kr^2}\left(\frac{dr}{d\lambda}\right)^2 - R^2 r^2\left(\frac{d\theta}{d\lambda}\right)^2 = 0$$

$$c^2\left(\frac{dt}{d\lambda}\right)^2 = \frac{R^2}{1-kr^2}\left(\frac{dr}{d\lambda}\right)^2 + R^2 r^2\left(\frac{d\theta}{d\lambda}\right)^2$$ 4.31

$$\frac{c^2}{R}\left(\frac{dt}{d\lambda}\right)^2 = \frac{R}{1-kr^2}\left(\frac{dr}{d\lambda}\right)^2 + Rr^2\left(\frac{d\theta}{d\lambda}\right)^2$$ 4.32

From (**4.27**), we have

$$c^2\frac{d^2 t}{d\lambda^2} + \dot{R}\left[\frac{R}{1-kr^2}\left(\frac{dr}{d\lambda}\right)^2 + Rr^2\left(\frac{d\theta}{d\lambda}\right)^2\right] = 0$$

Substituting equation (4.32), we have

$$c^2\frac{d^2 t}{d\lambda^2} + \frac{\dot{R}c^2}{R}\left(\frac{dt}{d\lambda}\right)^2 = 0$$

$$\frac{d^2t}{d\lambda^2}+\frac{\dot{R}}{R}\left(\frac{dt}{d\lambda}\right)^2=0$$

$$\frac{d}{d\lambda}\left(R\frac{dt}{d\lambda}\right)=R\frac{d^2t}{d\lambda^2}+\dot{R}\left(\frac{dt}{d\lambda}\right)^2$$

$$=R\left[\frac{d^2t}{d\lambda^2}+\frac{\dot{R}}{R}\left(\frac{dt}{d\lambda}\right)^2\right]=0$$

$$\frac{d}{d\lambda}\left(R\frac{dt}{d\lambda}\right)=0$$

$$\frac{dr}{d\lambda}\frac{d}{dr}\left(R\frac{dt}{d\lambda}\right)=0$$

$$or\ \text{since}\ \ \frac{dr}{d\lambda}\neq 0,\qquad \frac{d}{dr}\left(R\frac{dt}{d\lambda}\right)=0$$

And therefore

$R\frac{dt}{d\lambda}=h$ where h is a constant, not dependent on r

$$\frac{dt}{d\lambda}=\frac{h}{R}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 4.33$$

Substituting (**4.33)** and (**4.30**) into **(4.31**), have

$$c^2\left(\frac{h}{R}\right)^2=\frac{R^2}{1-kr^2}\left(\frac{dr}{d\lambda}\right)^2+R^2r^2\left(\frac{a}{R^2r^2}\right)^2$$

$$\frac{c^2h^2}{R^2}-\frac{a^2}{R^2r^2}=\frac{R^2}{1-kr^2}\left(\frac{dr}{d\lambda}\right)^2$$

$$\left(\frac{dr}{d\lambda}\right)^2=\frac{\left(1-kr^2\right)}{R^4}\left(c^2h^2-\frac{a^2}{r^2}\right)$$

$$=\frac{a^2\left(1-kr^2\right)\left(\frac{c^2h^2r^2}{a^2}-1\right)}{R^4r^2}$$

Setting $\alpha = \left(\frac{ch}{a}\right)^2$

$$\frac{dr}{d\lambda} = \frac{a\left(1-kr^2\right)^{\frac{1}{2}}\left(\alpha r^2 -1\right)^{\frac{1}{2}}}{R^2 r}$$

$$\frac{d\theta}{d\lambda}\frac{dr}{d\theta} = \frac{a\left(1-kr^2\right)^{\frac{1}{2}}\left(\alpha r^2 -1\right)^{\frac{1}{2}}}{R^2 r}$$

$$\frac{dr}{d\theta} = \frac{a\left(1-kr^2\right)^{\frac{1}{2}}\left(\alpha r^2 -1\right)^{\frac{1}{2}}}{R^2 r}\frac{d\lambda}{d\theta}$$

$$\frac{dr}{d\theta} = \frac{a\left(1-kr^2\right)^{\frac{1}{2}}\left(\alpha r^2 -1\right)^{\frac{1}{2}}}{R^2 r}\frac{R^2 r^2}{a}$$

$$\frac{dr}{d\theta} = r\left(1-kr^2\right)^{\frac{1}{2}}\left(\alpha r^2 -1\right)^{\frac{1}{2}}$$

$$\frac{dr}{r\left(1-kr^2\right)^{\frac{1}{2}}\left(\alpha r^2 -1\right)^{\frac{1}{2}}} = d\theta$$

$$\frac{dr}{r^3\left(\alpha - \frac{1}{r^2}\right)^{\frac{1}{2}}\left(\frac{1}{r^2}-k\right)^{\frac{1}{2}}} = d\theta$$

$$\frac{-\frac{1}{2}d\left(\frac{1}{r^2}\right)}{\left(\alpha - \frac{1}{r^2}\right)^{\frac{1}{2}}\left(\frac{1}{r^2}-k\right)^{\frac{1}{2}}} = d\theta$$

Setting $u = \frac{1}{r^2}$

$$\int \frac{du}{\left(\alpha - u\right)^{\frac{1}{2}}\left(u-k\right)^{\frac{1}{2}}} = -2\int d\theta \quad \text{...4.34}$$

Making the substitution

$$u = \alpha \cos^2 \phi + ks\text{in}^2 \phi ....4.35$$

$$du = [-2\alpha \cos\phi \sin\phi + 2k \cos\phi \sin\phi] d\phi$$

$$du = -2\cos\phi \sin\phi (\alpha - k) d\phi$$

$$\sqrt{(\alpha - u)(u - k)} = \sqrt{(\alpha - \alpha\cos^2\phi - k\sin^2\phi)(\alpha\cos^2\phi + k\sin^2\phi - k)}$$

$$= \sqrt{(\alpha - \alpha + \alpha\sin^2\phi - k\sin^{2\phi})(\alpha\cos^2\phi + k - k\cos^2\phi - k)}$$

$$= \sqrt{(\alpha - k)\sin^2\phi(\alpha - k)\cos^2\phi}$$

$$= (\alpha - k)\sin\phi\cos\phi$$

Equation (**4.34**) becomes

$$\int \frac{-2\cos\phi\sin\phi(\alpha - k)d\phi}{(\alpha - k)\cos\phi\sin\phi} - 2\int d\theta$$

$$\phi = \theta$$

From **(4.35)**

$$u = \alpha\cos^2\theta + k(1 - \cos^2\theta)$$

$$u = (\alpha - k)\cos^2\theta + k$$

$$\frac{1}{r^2} = (\alpha - k)\cos^2\theta + k$$

$$1 = (\alpha - k)r^2\cos^2\theta + kr^2$$

Using the substitutions

$$x = R\sin\chi\cos\theta$$

$$\text{y=Rsin}\chi\text{sin}\theta$$

$$\text{z=Rcos}\chi$$

And putting R to be constant in any particular epoch, we have

$$x^2 + y^2 = r^2$$

$$1 = (\alpha - k)x^2 + k(x^2 + y^2)$$
$$1 = \alpha x^2 + ky^2 \quad \text{...4.36}$$

These null geodesics are the same as the geodesics obtained from the massive particle.

Chapter 5

Discussion

5.1 The non-null geodesic

After determining the geodesics for the Robertson-Walker metric in (2+1)-dimensional spacetime, we found that these geodesics are all curves and can lie on surfaces. Thus, when the curvature parameter $k = 0$ we have a straight line which is a section of the Euclidean plane.

When the curvature parameter $k = +1$, we have great circles and ellipses. They are closed. When the curvature parameter $k = -1$ we have hyperbolae which describe the surface of a saddle. They are open.

These geodesics are curves because of the presence of gravity and curvature of the space. It is therefore not possible in the presence of gravity and hence curvature, to transform all geodesics into straight lines. Under such circumstances geodesics are intrinsically curved. As a result, spacetime itself is said to be curved by the distribution of mass and energy in it. Bodies like the earth are not made to move on curved orbits by gravity; instead, they follow the nearest thing to a straight path in a curved space, which is called a geodesic. For example, the surface of the sphere is a two-dimensional curved space. A geodesic on the sphere is called a great circle, and is the shortest path between two nearby points.

5.2 The null geodesic

The null geodesic is the path described by a photon. A photon is a quantum of light. Light rays too must follow geodesics in spacetime. Again, the fact that space is curved means that light no longer appears to travel in straight lines in space. So general relativity predicts that light should be bent by gravitational fields. For example, the theory predicts that the light cones of points near the sun would be slightly bent inward, on the account of the mass of the sun. This means that light from a distant star happened to pass near the sun would be deflected through a small angle, causing the star to appear in a different position to an observer on the earth (**figure 5.1**).

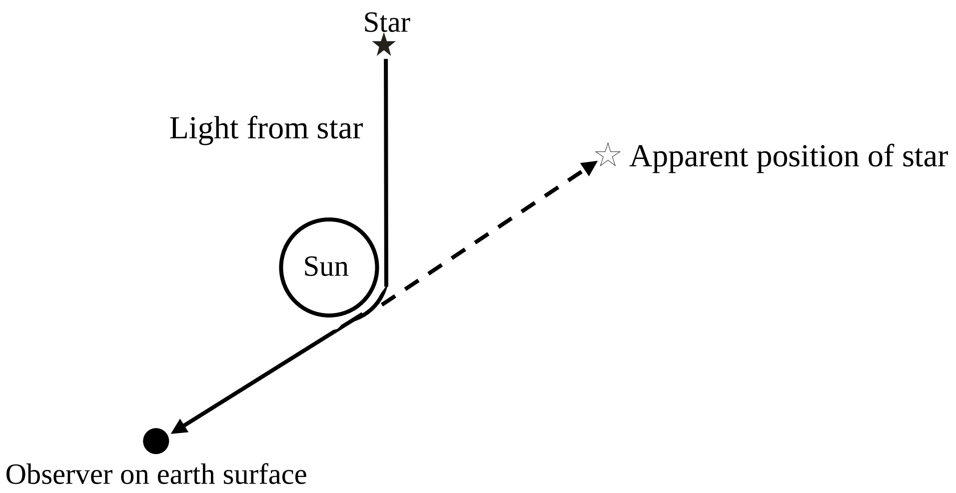

Figure 5.1 Shows deflection of light by the sun

The bending of a photon by a black hole

A black hole is a region of spacetime from which nothing, not even light, can escape because of its strong gravity.

The path of a photon from a distant source is bent by a black hole due to its strong gravity

The equations of the null geodesic (paths described by a photon) are given by

$$\alpha x^2 + ky^2 = 1$$

Thus, when $k = 0$ the photon with finite energy is absorbed by a black hole and therefore describes a straight line. When $k = +1$ and $\alpha = 1$, the photon describes a circular trajectory around a black hole. When $k = -1$ the photon is deflected outwards due to the strong gravity possessed by the black hole to describe a hyperbola

5.3 Conclusion

The geodesics determined in (2+1)-dimensional spacetime are the same as the geodesics in (3+1)-dimensional spacetime. This is due to the fact that the 2-dimensional surfaces on which these geodesics lie are embeddable in 3-dimensional coordinate space. The determination of the geodesics in (2+1)-dimensional spacetime are easier than in (3+1) - dimensional spacetime since the number of equations involved in (2+1)-dimensional spacetime are much smaller than those in (3+1) dimensional spacetime. This reason serve to buttress the viewpoint that certain problems in (3+1)-dimensional spacetime can be more easily solved by considering them in (2+1)-dimensional spacetime.

After constructing the 3-dimensional equivalent of the Robertson –Walker metric, we computed all the geodesics using the equation for the geodesics in general relativity. We found that these geodesics are surfaces of zero, positive, and negative curvatures. It was not surprising that, the geodesics in the plane and spherical surface were found to be a straight line and great circles respectively. What can apparently be considered to be new results found are the geodesics on the ellipsoidal surface and the surface of a saddle. These geodesics are ellipses and hyperbolae respectively.

Since all these geodesics are curves, it means that spacetime itself is curved and that bodies follow these geodesics as their trajectory.

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