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TWO DIFFERENT TOPOLOGIES ON THE DUAL OF A HILBERT SPACE AND THE MERITS OF EACH OF THEM

BY

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A THESIS
SUBMITTED TO THE BOARD OF POSTGRADUATE STUDIES, KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY (KNUST), KUMASI, GHANA, IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF MASTER OF SCIENCE (MSc) IN MATHEMATICS

AUGUST, 2009

DECLARATION

I declare that I have wholly undertaken the study report therein under supervision.

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This is a true account of the candidate's own work except references to other people's work which have been duly acknowledged.

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The student has satisfied all the requisite departmental requirements for this thesis to merit an MSc. Degree. I therefore permit him to present it for assessment.

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DEDICATION

This thesis is dedicated to my late Mother who through her belief in me, sweat to finance my education and most importantly her prayer saw me through the borders of the university successfully.

ACKNOWLEDGEMENT

I am deeply indebted and grateful to my supervisor Dr. S.A. Opoku, who braved all odds to get me to complete this work especially regarding his exclusive guidance, constructive and objective criticism provided during the execution of this work.

My sincere thanks also go to the entire staff of the Mathematics Department, K.N.U.S.T., for the motivation, encouragement and sound advice especially Dr. E. Prempeh, Rev. Dr. G.O. Lartey and Dr. E. Osei- Frimpong.

I cannot end without thanking my guardians, Mr and Mrs. Offei Quartey and my friend, Augustine Larweh Mahu who is more than a brother to me for their moral support.

I give thanks to Almighty God for seeing me through the entire MSc. Programme successfully.

ABSTRACT

In this work we considered two topologies of a Hilbert space l_2. The first one was its normed vector space topology, which is its natural topology. The second was its weak topology.

Among other properties we show that the closed unit ball S is not compact when l_2^* is given its topology as a Banach space. On the other hand S is compact when l_2^* is given its weak topology.

CONTENTS

CHAPTER ONE

INTRODUCTION

Topology and functional analysis are important areas of mathematics, the study of which will not only introduce you to new concepts and theorems but put into context old ones like continuous functions. However, to say just this is to understate the significance of topology and functional analysis. It is so fundamental that its influence is evident in almost every other branch of mathematics. This makes the study of topology relevant to all who aspire to be mathematicians whether their first love is (or will be) algebra, analysis, dynamics, industrial mathematics, mathematical biology, mathematical economics, mathematical finance, mathematical modeling, mathematical physics, mathematics of communication, number theory, numerical mathematics, operations research or statistics. In their study, we have basically the concepts which are usually implied or found in theorems. We therefore consider the theorems together with their proofs. It is in this regard that we have come to realize the need to have this research work.

Let R be the field of all real numbers.

Given a Banach space V over R. Let V^* be the dual of R. V^* is the space of all continuous linear functionals on V. There are several ways of defining a topology on V^*. In this thesis two such topologies will be discussed.

If V is a Banach space, the closed unit ball may not be compact. In a finite dimensional Banach space the closed unit ball is compact (by the Heine Borel theorem).

However in an infinite dimensional Banach space, when V^* is given the normed topology, the closed unit ball in V^* is not compact.

On the other hand when V^* is given its weak topology the closed unit ball in V^* is compact.

The space l_2 of sequences $\{x_n\}$ of real numbers for which $\sum_{n=1}^{\infty} x_n^2 < \infty$ will be used to illustrate this idea.

CHAPTER TWO

2 LITERATURE REVIEW

Here we define certain concepts which will be used in the course of this research work. They are those topological and analytical terms that must be understood before one can appreciate this research.

2.1 VECTOR SPACE

Many of the metric spaces which arise in analysis are endowed with a vector space structure, and the metrics are derived from norms related to this structure.

A vector space V **over a field** F consists of the set V, a mapping $(x, y) \to x + y$ of V^2 into V, and a mapping $(a, x) \to ax$ of $F \times V$ into V, such that

(a) V is an abelian group,

(b) $(a+b)x = ax + bx$ for each $a, b \in R, x \in V$,

(c) $a(x+y) = ax + ay$ for every $x, y \in V, a \in R$,

(d) $a(bx) = (ab)x$ for every $a, b \in R, x \in V$,

(d) $1x = x$ where $1 \in R$ and $x \in V$.

2.1.1 EXAMPLES OF VECTOR SPACES OVER R

(i) R itself is a vector space over R. In this case, $V = R$ is the real additive group and $F = R$ is the real field. The operation ax is multiplication for the real field.

ii) The set R^n of n-tuples of real numbers $x = (x_1, x_2, ..., x_n)$, with the standard operations $(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$ and $a(x_1, x_2, ..., x_n) = (ax_1, ax_2, ...ax_n)$, is a vector space.

(iii) H the space of all sequences $f : N \to R$ $such that$

$$\sum_{n=1}^{\infty} \{ f(n) \}^2 < \infty .$$

2.2 LINEAR INDEPENDENCE AND BASIS

Let V be a vector space over a field F

1. A non-empty subset A of V is said to be linearly independent over F if for every finitely many distinct elements $\{a_1, ..., a_n\}$ of A and scalars $\lambda_1, ..., \lambda_n \in F$ the condition $\sum_{j=1}^{n} \lambda_j a_j = 0$ implies $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$.

2.A non-empty subset D of V is said to be linearly dependent if is not linearly dependent. Thus D is linearly dependent over F , if , and only if , there exist finitely many distinct elements $d_1, ...d_q \in D$ and scalars $\lambda_1, ..., \lambda_q \in F$ such that $\sum_{j=1}^{q} \lambda_j d_j = 0$ and at least one of $\{\lambda_1, ...\lambda_q\}$ is not 0.

3.A non-empty subset S of V is said to span V if for every $x \in V$ there exist

finitely many elements $u_1,.......,u_m \in S$ and scalars $\gamma_1,.....,\gamma_m \in F$ such that

$x = \sum_{j=1}^{m} \gamma_j u_j$ .Thus a basis B of a vector space V is linearly independent subset of V that spans V .

2.3 FINITE DIMENSIONAL VECTOR SPACES

A vector space is said to be finite dimensional if there exist many

elements $b_1,......b_t \in V$ such that the $\{b_1,......,b_t\}$ spans V .

A non zero vector space V is said to be infinite dimensional if V is not finite

dimensional.

An example of an infinite dimensional vector space is l_2. Its basis are $e_1,....,e_n$.

2.4 DIMENSION

When the vector space V is finite dimensional, then there exist a unique positive

integer R such that every basis of V contains exactly R elements. In this case R is

called the dimension.

2.4.1 EXAMPLE

The vector space R^3 has $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ as a basis, and therefore we have

$\dim R^3 = 3$.

2.4.2 PROPOSITION

Every finite dimensional vector space over R is isomorphic in the sense of linear algebra to R^n.

2.5 LINEAR MAPPING

A mapping $T : X \to Y$ is called a linear mapping if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \text{ for all } x, y \in X \text{ and all scalars } \alpha, \beta.$$

If the linear space Y is replaced by the scalar field K, then the linear map T in the special case is called a linear functional on X

2.5.1 REMARK

Since linear functionals are special forms of linear maps, any result proved for linear maps also holds for linear functionals.

2.5.2 PROPOSITION

Let X and Y be two linear spaces over a scalar field K, and let $T : X \to Y$ be a linear map. Then

(i) $T(0) = 0$

(ii) The range of T, $R(T) = \{ y \in Y \mid Tx = y$ for some $x \in X \}$

is a linear subspace of Y

(iii) T is one-to-one if, and only if $T(0)=0$ implies $x=0$

(iv) If T is one-to-one, then T^{-1} exist on $R(T)$ and $T^{-1}:R(T)\to X$ is a linear map.

PROOF

(i) Since T is linear, we have $T(\alpha x)=\alpha T(x)$ for each $x\in X$ and each scalar α. take $\alpha=0$ and (i) follows immediately.

(ii) We need to show that for $y_1, y_2 \in R(T)$ and α, β scalars, $\alpha y_1+\beta y_2 \in R(T)$. Now, $y_1, y_2 \in R(T)$ implies that there exist $x_1, x_2 \in X$ such that $T(x_1)=y_1$, $T(x_2)=y_2$. Moreover, $\alpha x_1+\beta x_2 \in X$ (since X is a linear space).

Furthermore, by the linearity of T,

$T(\alpha x_1+\beta x_2)=\alpha T(x_1)+\beta T(x_2)=\alpha y_1+\mathrm{B}y_2$. Hence $\alpha y_1+\beta y_2 \in R(T)$, and so $R(T)$ is a linear subspace of Y.

(iii) $(\Rightarrow)$ Assume T is one-to-one. Clearly

$Tx = 0 \Rightarrow T(x) = T(0)$ since T is linear and so $T(0) = 0$. But T is one-to-one. So $x = 0$

$(\Leftarrow)$ Assume that whenever $Tu = 0$, then u must be 0. We want to prove that T is one-to-one. So let $Tx = Ty$. Then, $Tx - Ty = 0$ and by linearity of $T, T(x-y) = 0$. By hypothesis, $x - y = 0$ which implies $x = y$. Hence T is one-to-one.

(iv) Let $T : X \to Y$ be one-to-one. Then $T^{-1} : R(T) \to X$ exists . We prove that T^{-1} is also linear.

Let $\alpha, \beta \in R$ and $y_1, y_2 \in R(T)$. We know $R(T)$ is a linear subspace of Y. Hence $\alpha y_1 + \beta y_2 \in R(T)$. Let $x_1, x_2 \in X$ be such that $Tx_1 = y_1$, $Tx_2 = y_2$. We then have : $x_1 = T^{-1} y_1$, $x_2 = T^{-!} y_2$. Moreover,

$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 = \alpha y_1 + \beta y_2$ so that

$T^{-1}(\alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2 = \alpha T^{-1} y_1 + \beta T^{-1} y_2$. Thus T^{-1} is linear.

2.6 TOPOLOGICAL SPACE

Let X be a non-empty set. A collection T of subsets of X called a topology in X if these conditions are satisfied.

1. $X \in T$, that is X is in the collection T .
2. $\phi \in T$, that is ϕ is in the collection T .
3. $\bigcap_{j=1}^{n} G_j \in T$ for every collection $G_1,....,G_n$ of elements of T
4. $\bigcup_{\gamma \in \Gamma} G_\gamma \in T$ for every collection $\{G_\gamma \mid \gamma \in \Gamma\}$ of elements of T

Under such circumstances the ordered pair (X,T) is called a topological space.

When there is no ambiguity about the topology T , we shall simply say that X is a topological space.

If $G \in T$ then G is called an open set in the topological space (X,T).

If G is an open set in (X,T) and $a \in G$ then G is called an open neighborhood of a in (X,T).

2.6.1 THEOREM

Let X be a topological space. Then these two statements on a subset V of X are equivalent

(a) V is an open set in X

(b) for every $x \in V$ there exist an open set G_x in X such that $x \in G_x \subset V$

PROOF

Suppose (a) is true. If $x \in V$ let $G_x = V$ $x \in G_x \subset V$

Thus (a)$\Rightarrow$(b)

Next , suppose (b) is true. For each $x \in V$ choose an open set G_x in V such that $x \in G_x \subset V$ then $V = \bigcup_{x \in V} G_x$

It follows that V is open and follows that V is open in X .

2.6.2 DEFINITION

Let X be a topological space. A subset H of X is said to be closed if $X - H$ is open in X .

2.6.3 THEOREM

Let X be a topological space. Then

(1) ϕ is a closed set in X

(2) X is a closed set in X

(3) $\bigcup_{j=1}^{n} H_j$ is a closed set in X for every finite collection $H_1,....,H_n$ of closed set in X

(4) $\bigcap_{\gamma\in\Gamma} H_\gamma$ is a closed set in X for every collection $\{H_\gamma \mid \gamma \in \Gamma\}$ of closed sets in X.

PROOF

(1) $X-\phi = X$ and X is open in X. Hence ϕ is closed.

(2) $X - X = \phi$ and ϕ is open in X. Hence X is closed.

(3) $X - \bigcup_{j=1}^{n} H_j = \bigcap_{j=1}^{n} (X - H_j)$ by De Morgan's rule. $X - H_j$ is open for each $j \in \{1,...,n\}$. Therefore $X - \bigcup_{j=1}^{n} H_j$ is open in X. Thus $\bigcup_{j=1}^{n} H_j$ is closed.

(4) $X - \bigcap_{\gamma\in\Gamma} H_\gamma = \bigcup_{\gamma\in\Gamma} (X - H_\gamma)$ by De Morgan's rule. $X - H_\gamma$ is open in X *for all* $\gamma \in \Gamma$. Hence $X - \bigcap_{\gamma\in\Gamma} H_\gamma$ is open in X. Thus $\bigcap_{\gamma\in\Gamma} H_\gamma$ is closed

2.6.4 DEFINITION

Let X be a topological space. A collection F of subsets of X is called a *subbasis* of the topology of X if for every open set G in X and every $g \in G$ there exist finitely many elements $D_1, \ldots\ldots, D_n \in F$ such that $g \in \bigcap_{j=1}^{n} D_j \subset G$.

2.6.5 CONSTRUCTION OF SUBBASIS

Let X be a set. Suppose F is a non-empty collection of subset of X such that $X = \bigcup_{D \in F} D$. Let T be the set of all subsets G of X such that for every $a \in G$ there exist finitely many elements $D_1, \ldots\ldots, D_n \in F$ such that $a \in \bigcap_{j=1}^{n} D_j \subset G$.

Then T is a topology in X. T is the topology generated by F on X or F is said to be a subbasis of the topology of X.

2.7 NORM ON A VECTOR SPACE

Let V be a vector space over R. A norm $\|,\|$ on V is a real–valued function on V which satisfies these conditions:

1. $\|x\| \geq 0$ *for all* $x \in V$
2. *for every* $w \in V$, $\|w\| = 0$ *iff* $w = 0$, *the zero in* V
3. $\|\lambda x\| = |\lambda| \|x\|$ *for every* $x \in V$ *and* $\lambda \in R$
4. $\|x + y\| \leq \|x\| + \|y\|$ *for every pair* $x, y \in V$.

2.7.1 EXAMPLES

1. Let F be the field of all real numbers or the field of all complex numbers.

Then a norm is defined on F by the formula $\|z\| = |z| \quad \forall z \in F$

2. Given any positive integer n, note that F^n is a vector space over F.

We can define a norm $\|,\|$ on F^n by two possible methods

(a) $\|x\| = \max\{|x_1|,....,|x_n|\}$ if $x = (x_1,.....,x_n)$

(b) $\|x\| = \sqrt{\sum_{j=1}^{n} |x_j|^2}$

2.8 INNER PRODUCT ON A VECTOR SPACE

Let V be a vector space over field F. A mapping $\langle,\rangle : V \times V \to F$ is

called an inner product if these conditions are satisfied:

1. $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in V$

2. $\langle x, x\rangle \geq 0$ for all $x \in V$

3. *For* $x \in V$, $\langle x, x\rangle = 0$ *iff* $x = 0$, *the zero in* V

4. $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ *whenever* $\lambda \in F$ *and* $x, y \in V$

5. $\langle x + z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$ *for all* $x, y, z \in V$

A direct consequence of the fourth axiom is the following lemma.

2.8.1 LEMMA

Let $((V,\langle,\rangle)$ be an inner product space over F.

If $x,y\in V$ and $\lambda\in F$ then $\langle x,\lambda y\rangle=\overline{\lambda}\langle x,y\rangle$

PROOF

$$\langle x,\lambda y\rangle=\overline{\langle \lambda y,x\rangle}$$

$$=\overline{\lambda\langle y,x\rangle}$$

$$=\overline{\lambda}\,\overline{\langle y,x\rangle}$$

$$=\overline{\lambda}\langle x,y\rangle$$

Hence the proof.

2.9 METRIC SPACE

Let Y be a non–empty set. A real–valued function d on $Y\times Y$ is called a metric on Y if these four conditions are satisfied:

1. $d(x,y)\geq 0$ *for every pair* $x,y\in Y$
2. *for every* $x,y\in Y$, $d(x,y)=0$ *iff* $x=y$
3. $d(x,y)=d(y,x)$ *for every pair* $x,y\in Y$
4. *for every* $x,y,z\in Y$, $\quad d(x,y)\leq d(x,z)+d(z,y)$

The ordered pair (Y,d) is called a metric space.

2.9.1 EXAMPLE

Let $d : R \times R \to R$ *be defined by*

$$d(x,y) = \frac{|x-y|}{1+|x-y|}\text{ , } for\ every\ x,y \in R$$

Then d is a metric.

PROOF

We want to show that d satisfies the four axioms of a metric space

1. $For\ every\ x,y \in R$

$$d(x,y) = \frac{|x-y|}{1+|x-y|} \geq 0 \quad \text{by definition}$$

Axiom 1 is satisfied.

2. $d(x,y) = \dfrac{|x-y|}{1+|x-y|}$

$$= \frac{|-1||(y-x)|}{1+|-1||y-x|}$$

$$d(x,y) = \frac{|y-x|}{1+|y-x|}$$

$$= d(y,x)$$

Axiom 2 is satisfied.

3. If $d(x,y) = \dfrac{|x-y|}{1+|x-y|} = 0$ then

$$|x-y|=0$$

$$\Rightarrow x-y=0$$

$$x=y$$

Conversely,

If $x=y, then$

$$d(x,y)=\frac{|x-y|}{1+|x-y|}=\frac{|y-y|}{1+|y-y|}=0$$

Axiom 3 is satisfied.

4. For every $x,y,z\in R$,

$$|x-y|\le|x-z|+|z-y|$$

Therefore $\dfrac{|x-y|}{1+|x-y|}\le\dfrac{|x-z|+|z-y|}{1+|x-z|+|z-y|}$

$$\le\frac{|x-z|}{1+|x-z|}+\frac{|z-y|}{1+|z-y|}$$

$$\Rightarrow d(x,y)\le d(x,z)+d(z,y)$$

Axiom 4 is satisfied. Therefore d is a metric on Y.

2.9.2 DEFINITION

Define $d : H \times H \to R$ by

$d(f,g) = \sqrt{\sum_{n=1}^{\infty}\{f(n)-g(n)\}^2}$. Then d is a metric on H

PROOF

(i) For every pair $f, g \in H$ $d(f,g) = d(g,f)$ *by definition*

(ii) For every pair $f, g \in H$ $d(f,g) \geq 0$ *by definition*

In fact, $d(f,g) \geq |f(1)-g(1)| \geq 0$

(iii) For every $f, g \in H$ $d(f,g) = 0$

$$\Leftrightarrow \sum_{n=1}^{\infty}\{f(n)-g(n)\}^2 = 0$$

$$\Leftrightarrow f(n)-g(n) = 0 \ \textit{for every}\ n \geq 1$$

$$\Leftrightarrow f = g$$

(iv) Finally, if $f, g, h \in H$ then for every positive integer n

$$\{f(n)-g(n)\}^2 = \{f(n)-h(n)+h(n)-g(n)\}^2$$

$$=\{f(n)-h(n)\}^2 + 2\{f(n)-h(n)\}\{h(n)-g(n)\} + \{h(n)-g(n)\}^2$$

Hence by the Schwarz's inequality

$$\{d(f,g)\}^2 \leq \{d(f,h)\}^2 + 2d(f,h)d(h,g) + \{d(h,g)\}^2$$

It follows that $d(f,g) \leq d(f,h) + d(h,g)$

2.9.3 DEFINITION AND NOTATION

Given that (Y,d) is a metric space, let a collection T_d of subset of Y be defined as follows:

$G \in T_d$ if, and only if for every $a \in G$ there exist a positive real number δ such that $\{y \in Y \mid d(y,a) < \delta\} \subset G$.

Then T_d is a topology in Y. [T_d is called the topology induced on Y by d]

We proceed to show that T_d is a topology in Y:

(1) if $a \in Y$ then $\{y \in Y \mid d(a,y) < \delta\} \subset Y$.

Hence $Y \in T_d$.

(2) $\phi \in T_d$ because ϕ is empty.

(3) Suppose $G_1, \ldots, G_n$ are finitely many elements of T_d. If $b \in \bigcap_{j=1}^{n} G_j$ choose for each $j \in \{1,\ldots,n\}$ a positive real number $\delta_j > 0$ such that

$\{y \in Y \mid d(y,b) < \delta_j\} \subset G_j$.

Let $\delta = \min\{\delta_1,\ldots,\delta_n\}$. Then $\{y \in Y \mid d(y,b) < \delta\} \subset \bigcap_{j=1}^{n} G_j$.

Thus $\bigcap_{j=1}^{n} G_j \in T_d$

(4) Suppose Γ is a set and for every $\gamma \in \Gamma \quad G_\gamma \in T_d$. If $w \in \bigcup_{\gamma\in\Gamma} G_\gamma$ choose

$\alpha \in \Gamma$ such that G_α and a positive real number τ such that

$\{y \in Y \mid d(y,w) < \tau\} \subset \bigcup_{\gamma\in\Gamma} G_\gamma$. Hence $\bigcup_{\gamma\in\Gamma} G_\gamma \in T_d$.

These show that T_d is a topology in Y.

2.10 THE RELATION BETWEEN METRIC, NORM AND INNER PRODUCT

2.10.1 THEOREM

Given a normed vector space $(V, \|,\|)$ *over* F, define

$$d : V \times V \to R \text{ by}$$

$$d(x,y) = \|x - y\|$$

Then $d(x,y)$ is a metric on V. d is called the metric induced on V by $\|,\|$

PROOF

We show that d satisfies the four axioms of a metric.

1 For any arbitrary $x, y \in V$.

$$\|x - y\| = \|(-1)(y - x)\|$$

$$= |-1| \|y - x\|$$

Therefore $d(x,y) = d(y,x)$

Axiom 1 is satisfied.

1. For every $x, y \in V$,

$$d(x, y) = \|x - y\| \geq 0$$

Axiom 2 is satisfied.

2. For every $x, y \in V$, $d(x, y) = 0$

$$\Leftrightarrow \|x - y\| = 0$$

$$\Leftrightarrow x - y = 0$$

$$\Leftrightarrow x = y$$

Axiom 3 is satisfied.

3. $If\ x, y, z \in V,\ then$

$$\|x - y\| = \|(x - z) + (z - y)\|$$

$$\leq \|x - z\| + \|z - y\| \ by\ the\ Triangle\ Inequality$$

Therefore $d(x, y) \leq d(x, z) + d(z, y)$

Axiom 4 is satisfied.

Thus d is a metric on V.

d is called the metric induced on V by the norm, $\|,\|$.

2.10.2 THEOREM

Given an inner product space $(V, \langle , \rangle)$ define $\|,\|$ on V by

$$\|x\| = \sqrt{\langle x, x \rangle} \quad for\, all\, x \in V$$

Then $\|,\|$ is a norm on V .

PROOF

1. By definition, $\|x\| \geq 0$.

2. $For\ x \in V, \|x\| = 0 \Leftrightarrow \sqrt{\langle x, x \rangle} = 0$

$$\Leftrightarrow \langle x, x \rangle = 0$$

$$\Leftrightarrow x = 0$$

3. $Let\ \lambda \in F\ and\ x \in V, then$

$$\|\lambda x\|^2 = \langle \lambda x, \lambda x \rangle$$

$$= |\lambda|^2 \langle x, x \rangle$$

$$Therefore\ \|\lambda x\| = |\lambda| \|x\|$$

4. *Let* $x, y \in V$, then

$$\|x + y\|^2 = \langle x + y, x + y \rangle$$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$

$$= \|x\|^2 + 2\operatorname{Re}\langle x, y \rangle + \|y\|^2$$

$$\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2$$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2$$

Therefore $\|x+y\| \leq \|x\| + \|y\|$

2.11 THEOREM(Schwarz's Inequality)

Let $(V, \langle , \rangle)$ be an inner product space over the field F. Then

$|\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle}$ for every pair $x, y \in V$

PROOF

Given $x, y \in V$ let $A = \langle x, x\rangle, B = \langle x, y\rangle$ and $D = \langle y, y\rangle$. Then we want to show that $|B| \leq \sqrt{A}\sqrt{D}$

If $A = 0,$ then $x = 0$ and so $B = 0$. In this case we have $|B| = 0 = \sqrt{A}\sqrt{D}.$

If $A > 0$ let $\lambda = \dfrac{B}{A}$.

$$0 \le \langle \lambda x - y, \lambda x - y \rangle$$

$$= \langle \lambda x, \lambda x \rangle - \langle \lambda x, y \rangle - \langle y, \lambda x \rangle + \langle y, y \rangle$$

$$= |\lambda|^2 A - \lambda \langle x, y \rangle - \bar{\lambda} \langle y, x \rangle + D$$

$$= |\lambda|^2 A - \frac{B\bar{B}}{A} - \frac{B\bar{B}}{A} + D$$

$$0 \le D - \frac{|B|^2}{A} \Rightarrow \frac{|B|^2}{A} \le D$$

Then

Thus $|B|^2 \le AD$. Hence $|B| \le \sqrt{A}\sqrt{D}$, which is Schwarz's inequality.

Schwarz's inequality is equivalently $|\langle x, y \rangle| \le \|x\| \|y\|$.

2.12 NORMED TOPOLOGY

Given a normed vector space $(V, \|,\|)$. Let a collection T of subset of V be defined as follows:

$G \in T$ if and only if for every $a \in G$ there exist a positive real number δ such that $\{x \in V \mid \|x - a\| < \delta\} \subset G$. Then T is a topology in V. T is called the topology induced on V by the norm. This is the same as the topology induced on V by the metric d. Where $d(x, y) = \|x - y\|$.

Henceforth when we say that $(V, \|,\|)$ is a vector space, we mean V is a vector space, $\|,\|$ is a norm on V and there is a topology T induced on V by the norm.

2.13 PYTHAGORAS THEOREM

If $(V, \langle,\rangle)$ is an inner product space $a, b \in V$ and $\langle a, b\rangle = 0$ then

$$\|a+b\|^2 = \|a-b\|^2 = \|a\|^2 + \|b\|^2 .$$

PROOF

$$\|a+b\|^2 = \langle a+b, a+b\rangle = \langle a,a\rangle + \langle a,b\rangle + \langle b,a\rangle + \langle b,b\rangle$$

$$= \|a\|^2 + \|b\|^2$$

And

$$\|a-b\|^2 = \langle a-b, a-b\rangle = \langle a,a\rangle - \langle a,b\rangle - \langle b,a\rangle + \langle b,b\rangle = \|a\|^2 + \|b\|^2 .$$

2.14 PARALLELOGRAM LAW

Let V be an inner product space. Then for arbitrary $x, y \in V$,

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

2.15 BOUNDEDNESS

A function $f \in \hbar(X, R)$ is said to be bounded if and only if there exists a positive real number M such that $\left|f(x)\right| \leq M$ for every $x \in X$.

We say f that is bounded below if there exist a real number ω such that

$\omega \leq x \ for\ all\ x \in X.$

Under such circumstance ω is called a lower bound of X.

We say that f is bounded above if there exist a real number α such that

$x \leq \alpha \ for\ all\ x \in X$.

Under such circumstance α is called an upper bound of X .

2.15.1 DEFINITION

Let S be a set of real numbers.

We say that S is bounded if there exist a positive real number M such that

$\left|x\right| \leq S \ for\ all\ x \in S.$

2.15.2 THEOREM

These two statements on a non-empty set S of real numbers are equivalent

i. S is bounded.

ii. S is bounded above and bounded below.

PROOF

Suppose i is true.

Choose a positive real number M such that $|x| \leq M \;\; for\ all\ x \in S.$

Then $x \leq |x| \leq M$ and

$$-x \leq |x| \leq M \;\; for\ all\ x \in S$$

$$x \leq M \;\; for\ all\ x \in \delta \Rightarrow S\ is\ bounded\ above$$

And

$$-x \leq M \;\; for\ all\ x \in \delta \Rightarrow -M \leq x \;\; for\ all\ x \in S\ and\ S\ is\ bounded\ below.$$

Thus $i \Rightarrow ii$

Suppose ii is true

Choose real numbers u, ω such that

$$x \leq u \;\; for\ all\ x \in S\ and\ \omega \leq x \;\; for\ all\ x \in S$$

Then $x \leq u \leq |u| \;\; for\ all\ x \in S$

Also $-x \leq -\omega \leq |\omega| \;\; for\ all\ x \in S.$

Let $M = |u| + |\omega|$

Then $x \le M$ *and* $-x \le M$ *for all* $x \in S$.

Therefore $|x| \le M$ *for all* $x \in S$.

Thus S is bounded and so $ii \Rightarrow i$.

Let X be a metric space with metric d, and let $A \subset X$.

If x is a point of X, then the distance from x to A is defined by

$$d(x, A) = \inf\left\{d\left(x, a\right) : a \in A\right\};$$

That is, it is the greatest lower bound of the distance from x to the points of A.

The diameter of the set A is defined by

$$d(A) = \sup\left\{d\left(a_1, a_2\right) : a_1, a_2 \in A\right\}$$

The diameter of A is thus the least upper bound of the distances between pairs of its points. A is said to have finite diameter or infinite diameter according as $d(A)$ is a real number or $\pm\infty$.

We observe that the empty set has infinite diameter, since $d(\phi) = -\infty$.

A bounded set is one whose diameter is finite.

A mapping of non-empty set into a metric space is called a bounded mapping if its range is a bounded set.

2.15.3 EXAMPLE

If $Y = \left\{ \dfrac{x}{1+x} : x \in R\, and\ x > 0 \right\}$

a. Prove that Y is bounded.

b. Find $\inf Y$.

c. Find $\sup Y$.

SOLUTION

a. Let $y \in Y$ then $y = \dfrac{x}{1+x}$ *where x is a* R^{+}

$$\Leftrightarrow y + yx = x$$

$$\Leftrightarrow y = x - yx$$

$$\Leftrightarrow y = x(1-y)$$

$$\Leftrightarrow x = \frac{y}{1-y}$$

First of all $x > 0\ and\ \dfrac{1}{1+x} > 0$

$$\therefore y > 0.$$

Also $1 - \dfrac{x}{1+x} = \dfrac{1+x-x}{1+x}$

$$= \frac{1}{1+x} > 0$$

$$y = x \cdot \frac{1}{1+x}$$

$$\therefore 1 > \frac{x}{1+x} > 0$$

$$\therefore 0 < y < 1 \ \ if \ \ y \in Y.$$

Then Y is bounded above and Y is bounded below.

Hence Y is bounded.

b. If η is a real number such that $0 < \eta < 1$

Then the equation

$$\eta = \frac{\alpha}{1+\alpha}$$

$$\Leftrightarrow \eta(1+\alpha) = \alpha$$

$$\Rightarrow \alpha = \frac{\eta}{1-\eta}$$

Therefore η cannot be a lower bound of Y .

Hence $0 = \inf \ Y.$

$$0 < \tfrac{1}{2}(1+\eta) < 1$$

Similarly η cannot be an upper bound of Y

$\therefore \ \sup Y = 1$

2.16 BOUNDED LINEAR MAPPINGS

Let X and Y be normed linear spaces over the scalar field, K, and let $T: X \to Y$ be a linear map. Then T is said to be bounded if there exists some constant $k \geq 0$ such that for each $x \in X$ $\|T(x)\| \leq k\|x\|$.

2.16.1 THEOREM

Let $T: X \to Y$ be a linear mapping where X, Y are normed vector spaces.

Then these three statements are equivalent

(1) T is continuous at 0

(2) T is bounded

(3) T is uniformly continuous

PROOF

Suppose (1) is true.

Choose a positive real number δ such that for $x \in X$

$$\|x - 0\| < \delta \Rightarrow \|T(x) - T(0)\| < 1$$

That is $\|x\| < \delta \Rightarrow \|T(x)\| < 1$

Now suppose $z \in X$ and $z \neq 0$

Then $\left\| \dfrac{\frac{1}{2}\delta z}{\|z\|} \right\| = \frac{1}{2}\delta \dfrac{\|z\|}{\|z\|} = \frac{1}{2}\delta < \delta$

And so $\left\| T\left(\frac{\frac{1}{2}\delta z}{\|z\|} \right) \right\| < 1 \Rightarrow \frac{\frac{1}{2}\delta}{\|z\|}\left\| T(z) \right\| < 1 \Rightarrow \left\| T(z) \right\| < \frac{2}{\delta}\|z\|$

Let $k = \frac{2}{\delta}$

Then for every $v \in X \quad \left\| T(v) \right\| \le k\|v\|$

Therefore T is bounded. Thus $1 \Rightarrow 2$

Suppose (2) is true. Given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{k}$

Then for $z, v \in X \quad \|z - v\| < \frac{\varepsilon}{k}$

$\Rightarrow \left\| T(z) - T(v) \right\| = k\left\| T(z - v) \right\| < \varepsilon$

Therefore T is uniformly continuous. Thus $2 \Rightarrow 3$

$3 \Rightarrow 1$ by definition.

2.16.2 THEOREM

If X and Y are normed spaces, then $B(X, Y)$ the set of all bounded linear mappings from X into Y is a normed space with norm defined by

$$\|T\| = \sup_{\|x\|=1} \left\| T(x) \right\| \qquad 1$$

PROOF

We will only show that norm 1 satisfies the triangle inequality.

For all $T_1, T_2 \in B(X, Y)$ and every $x \in X$ such that $\|x\| = 1$

We have

$$\|T_1(x) + T_2(x)\| \le \|T_1(x)\| + \|T_2(x)\|$$

$$\le \sup_{\|x\|=1} \|T_1(x)\| + \sup_{\|x\|=1} \|T_2(x)\|$$

$$= \|T_1\| + \|T_2\|$$

Hence

$$\sup_{\|x\|=1} \left(\|T_1(x) + T_2(x)\| \right) = \|T_1 + T_2\| \le \|T_1\| + \|T_2\|$$

2.17 COMPACTNESS

A subset of a topological space is said to be compact if every open cover has a finite subcover. The definition of compactness can be obtained from the Classical Heine-Borel Theorem. It is stated below:

2.17.1 HEINE-BOREL THEOREM

Let n be a positive integer. Give R^n its Euclidean norm $\|,\|$ defined by

$\|x\| = \sqrt{\sum_{j=1}^{n} x_j^2}$ where $x = (x_1, ..., x_n)$. Then a subset B of R^n is compact if and only if B is closed and bounded.

2.17.2 EXAMPLES

1. By the Heine-Borel Theorem, every closed and bounded interval $[a, b\ ]$ on the real line R is compact.
2. Let A be any finite subset of a topological space X, say $A = \{a_1, ..., a_m\}$.Then A is necessarily compact. For if $G = \{G_i\}$ is an open cover of A, then each point in A belongs to one of the members of G, say $a_1 \in G_{i_1}, ..., a_m \in G_{i_m}$.

   Accordingly, $A \subset G_{i_1} \bigcup G_{i_2} \bigcup .... \bigcup G_{i_m}$.

   Since a set A is compact iff every open cover of A contains a finite subcover, we only have to exhibit one open of A with no finite to prove that A is not compact.

2.18 BOLZANO- WEIERSTRASS PROPERTY OF COMPACT SET IN A NORMED VECTOR SPACE.

If A is a compact set in a normed vector space $(V,\|,\|)$ and a_n is a sequence in A, then there exist $b \in A$ and the subsequence a_{n_k} of a_n such that $a_{n_k} \to b$ as $k \to \infty$.

We hope that eventually for infinite dimensional Hilbert space, we will be able to give an example of a closed and bounded subset which is not compact.

2.19 CONVERGENT SEQUENCE

A sequence $\{X_n\}$ in a normed vector space $(V,\|,\|)$ is said to *converge* if there is a point $p \in V$ with the following property: For every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $\|X_n - p\| < \varepsilon$

In this case we also say that $\{X_n\}$ converges to p, or that p is the limit of $\{X_n\}$, and we write $X_n \to p$, or $\lim_{n\to\infty} X_n = p$.

More briefly, a sequence $\{X_n\}$ converges to an element $p \in V$ if $\lim_{n\to\infty} \|x_n - p\| = 0$

2.20 CAUCHY SEQUENCE

Let $(V, \|,\|)$ be a normed vector space. A sequence $\{X_n\}$ in V is called a Cauchy sequence in $(V, \|,\|)$ if to every positive real number ε, there corresponds a positive integer P such that $m > P$ and $n > P$ implies $\|X_m - X_n\| < \varepsilon$.

More briefly, $\{X_n\}$ is a Cauchy sequence if $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$.

2.20.1 THEOREM

If $\{x_n\}$ is a convergent sequence in a normed vector space $(V, \|,\|)$ then $\{x_n\}$ is a Cauchy sequence in $(V, \|,\|)$.

PROOF

Let $a = \lim_{n\to\infty} X_n$

Given a positive real number ε. Choose a positive integer P such that

$\|x_n - a\| < \dfrac{\varepsilon}{2}$ $\mathit{for\ all}\ n \ge P$ .Then

$\|x_m - x_n\| \le \|x_m - a\| + \|a - x_n\| < \varepsilon$ $\mathit{for\ all}\ m \ge P\ \mathit{and}\ \mathit{for\ all}\ n \ge P$.

2.21 COMPLETENESS

A normed vector space $\left(V,\|,\|\right)$ is said to be *complete* if every Cauchy sequence in $\left(V,\|,\|\right)$ is a convergent sequence in $\left(V,\|,\|\right)$.

2.22 BANACH SPACE

A normed vector space $\left(V,\|,\|\right)$ is called a Banach space if every Cauchy sequence in it is a convergent sequence.

2.22.1 THEOREM

If X is a normed space and Y is a Banach space, then $B\left(X,Y\right)$ is defined by

$\|T\| = \sup_{\|x\|=1}\|T\left(x\right)\|$ is a Banach space.

PROOF

We need to show that $B\left(X,Y\right)$ is complete.

Let $\{T_n\}$ be a Cauchy sequence in $B\left(X,Y\right)$, and let x be an arbitrary element of T_1. Then

$$\|T_m\left(x\right)-T_n\left(x\right)\| \le \|T_m - T_n\|\|x\| \to 0 \; as\, m,n \to \infty$$

Which shows that $\{T_n\}$ is a Cauchy in Y.

By completeness of Y, there is a unique element $y \in Y$ such that $T_n \to y$

Since x is an arbitrary element of X , this defines a mapping T from X into Y :

$$\lim_{n\to\infty} T_n(x) = T(x)$$

We will show that $T \in B(X,Y)$ and $\|T_n - T\| \to 0$

Clearly, T is a linear mapping. Since Cauchy sequences are bounded, there exist a constant M such that $\|T_n\| \le M \quad for\, all\, n \in \mathbb{N}$. Consequently

$$\|T(x)\| = \left\|\lim_{n\to\infty} T_n(x)\right\| = \lim_{n\to\infty} \|T_n(x)\| \le M\,\|x\|$$

Therefore, T is bounded and thus $T \in B(X,Y)$.

It remains to show that

$\|T_n - T\| \to 0$. Let $\varepsilon > 0$ and let k be such that

$\|T_m - T_n\| < \varepsilon \;\; for\, every\, m,n \ge k$. If $\|x\| = 1$ and $m,n \ge k$,

then

$\|T_m(x) - T_n(x)\| \le \|T_m - T_n\| < \varepsilon$.

By letting $n \to \infty\ (m\ remains\ fixed\ )$,

we obtain

$\|T_m(x) - T(x)\| < \varepsilon\ for\, every\, m \ge k\, and\, every\, x \in X$

With $\|x\| = 1$. This means that $\|T_m - T\| < \varepsilon \;\; for\, all\, m > k$, which complete the proof.

2.22.2 DEFINITION

Given a Banach space V over R. Let V^* be the *dual* of R. That is V^* is the space of all continuous linear functionals on V.

2.22.3 PROPOSITION

$f \in V^*$ if, and only if $f : V \to R$ is a continuous linear functional.

2.23 HILBERT SPACE

A Banach space $\left(V, \|,\|\right)$ is called a Hilbert space if the norm, $\|,\|$ is induced by an inner product $\langle , \rangle$ on V.

Alternatively: Given an inner product space $\left(V, \langle , \rangle\right)$, let $\|,\|$ be the norm induced on V by the inner product $\langle , \rangle$. Then V is called a Hilbert space if and only if, every Cauchy sequence in V is a convergent sequence in V. Thus a complete inner product space is called a Hilbert space.

2.24 SOME USEFUL EXAMPLES

1.Given a positive integer n define $\langle , \rangle$ on R^n by $\langle x, y \rangle = \sum_{j=1}^{n} x_j y_j$ if $x = \left(x_1, ..., x_n\right)$ and $y = \left(y_1, ..., y_n\right)$

Then $\langle , \rangle$ is an inner product on R^n. For each $x = \left(x_1, ..., x_n\right) \in R^n$

$$\langle x, x \rangle = \sum_{j=1}^{n} x_j^{\,2} \quad \text{and} \quad \|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{j=1}^{n} x_j^{\,2}}$$

The Euclidean metric d on R^n is defined by

$$d\left(x,y\right)=\left\|x-y\right\|=\sqrt{\sum_{j=1}^{n}\left(x_j-y_j\right)^2} \text{ of } x=\left(x_1,...,x_n\right) \text{ and } y=\left(y_1,....,y_n\right)$$

Under such circumstances Schwartz's inequality is

$$\left|\sum_{j=1}^{n}x_j y_j\right|\le\sqrt{\sum_{j=1}^{n}x_j{}^2}\sqrt{\sum_{j=1}^{n}y_j{}^2}$$

and it is often referred to as Cauchy-Schwarz inequality.

2- The Hilbert space H : Let H be the set of all mapping such that

$$\Psi : N\rightarrow R \ such\, that \sum_{n=1}^{\infty}\left|\Psi\left(n\right)\right|^2 \text{ is convergent}$$

Where N is the set of all positive integers and R is the set of all real numbers.

If $f,g\in H$ then for every positive integer k Cauchy-Schwarz inequality is

$$\left|\sum_{n=1}^{k}f\left(n\right)g\left(n\right)\right|\le\sqrt{\sum_{n=1}^{k}\left|f\left(n\right)\right|^2}\sqrt{\sum_{n=1}^{k}\left|g\left(n\right)\right|^2}$$

$$\left|\sum_{n=1}^{k}f\left(n\right)g\left(n\right)\right|\le\sqrt{\sum_{n=1}^{\infty}\left|f\left(n\right)\right|^2}\sqrt{\sum_{n=1}^{\infty}\left|g\left(n\right)\right|^2}$$

Similarly

$$\sum_{n=1}^{\infty}\left|f\left(n\right)\right|\left|g\left(n\right)\right|\le\sqrt{\sum_{n=1}^{\infty}\left|f\left(n\right)\right|^2}\sqrt{\sum_{n=1}^{\infty}\left|g\left(n\right)\right|^2}$$

Let $k\rightarrow\infty$

Then $\displaystyle\sum_{n=1}^{\infty}\left|f\left(n\right)\right|\left|g\left(n\right)\right|\le\sqrt{\sum_{n=1}^{\infty}\left|f\left(n\right)\right|^2}\sqrt{\sum_{n=1}^{\infty}\left|g\left(n\right)\right|^2}<\infty$

The conclusion is that $\sum_{n=1}^{\infty} f(n)g(n)$ is absolutely convergent. That implies

$\sum_{n=1}^{\infty} f(n)g(n)$ is convergent and $\left|\sum_{n=1}^{\infty} f(n)g(n)\right| \leq \sqrt{\sum_{n=1}^{\infty}|f(n)|^2}\sqrt{\sum_{n=1}^{\infty}|g(n)|^2}$

Now if $f, g \in H$

Then $\sum_{n=1}^{\infty}|f(n)+g(n)|^2 = \sum_{n=1}^{\infty}|f(n)|^2 + 2\sum_{n=1}^{\infty}|f(n)g(n)| + \sum_{n=1}^{\infty}|g(n)|^2 < \infty$

Thus $f+g \in H$ if both of f, g are in H. This make H a vector space over R.

Next an inner product $\langle , \rangle$ is defined on H by $\langle f, g\rangle = \sum_{n=1}^{n} f(n)g(n)$

Finally the norm induced on H by this inner product is given by the formular

$$\|f\| = \sqrt{\sum_{n=1}^{\infty}|f(n)|^2}.$$

2.25 CONTINUOUS FUNCTIONS

Let $(V, \|,\|)$ be a normed vector space and $f: V \to R$ a real-valued function on V. Given $a \in V$, we say that f is continuous at a if to every positive real number ε there corresponds a positive real number δ such that

$\|f(x) - f(a)\| < \varepsilon$ for every $x \in X$ such that $\|x-a\| < \delta$.

2.25.1 EXAMPLES

1. Constant mapping

   Let X be a non empty set of real numbers and k a constant real number define $f : S \to R$ by $f(x) = k$ *for all* $x \in S$

   Then f is continuous at every point on f

PROOF

Let $c \in S$. Given $\varepsilon > 0$ Let $\delta = 1$

Then

$$\left\| f(x) - f(c) \right\| = \left\| k - k \right\| = 0 < \varepsilon$$

$$\text{for all } x \in S \text{ such that } \left\| x - c \right\| < \delta$$

2. Define $I : R \to R$ *by* $I(x) = x$ *for all* $x \in R$

Then I is the identity mapping of R onto itself.

I is uniformly continuous on R and so I is continuous.

PROOF

Given a positive real number ε let

$\delta = \varepsilon$ .Then $\left\| I(x) - I(z) \right\| = \left\| x - z \right\| < \varepsilon$.

For every pair $x, z \in R$ such that $\left\| x - z \right\| < \delta$.

Therefore I is uniformly continuous on R .

3. If f is continuous at c then $\left\| f \right\|$ is also continuous at c .

PROOF

If X_n is a sequence in S such that $X_n \to c$ as $n \to \infty$

Then $f(X_n) \to f(c)\ as\ n \to \infty$

Hence $\left\| f(x_n) \right\| \to \left\| f(x) \right\| as\ n \to \infty$

4. If $f(x) \neq 0$ for every $x \in S$ and f is continuous at c

Then $\dfrac{1}{f}$ is also continuous at c .

PROOF

Let X_n be a sequence in S such that $X_n \to c \text{ as } n \to \infty$

Then $f\left(X_n\right) \to f\left(c\right) \text{ as } n \to \infty$

Hence $\dfrac{1}{f\left(X_n\right)} \to \dfrac{1}{f\left(c\right)} \text{ as } n \to \infty$

Therefore $\dfrac{1}{f}$ is continuous at c.

5. If f is continuous at c *and* λ is a constant real number

then λf is continuous at c.

PROOF

Suppose X_n is a sequence in S such that $X_n \to c \text{ as } n \to \infty$

Then $f\left(X_n\right) \to f\left(c\right) \text{ as } n \to \infty$

That implies, $\lambda f\left(X_n\right) \to \lambda f\left(c\right) \text{ as } n \to \infty$

Therefore λf is continuous at c.

2.25.2 THEOREM

If $f \; and \; g$ are real-valued functions on S such that f is continuous at c and g is continuous at c then

- $f+g$ is continuous at c
- fg is continuous at c

PROOF

Let X_n be a sequence in S such that $X_n \to c \; as \; n \to \infty$

Then $f\left(X_n\right) \to f\left(c\right) as \; n \to \infty$ while

$g\left(X_n\right) \to g\left(c\right) as \; n \to \infty$

Hence

i. $f\left(X_n\right) + g\left(X_n\right) \to f\left(c\right) + g\left(c\right) as \; n \to \infty$ and

ii. $f\left(X_n\right) g\left(X_n\right) \to f\left(c\right) g\left(c\right) as \; n \to \infty$

Thus $f+g$ is continuous at c and fg is continuous at c.

2.25.3 THEOREM

Let (X, d) and (Y, ρ) be metric spaces, $a \in X$, and $f : X \to Y$ a map.

Then these two statements are equivalent

(1) f is continuous at a

(2) if $a_n \to a$ as $n \to \infty$, then $f(a_n) \to f(a)$ as $n \to \infty$

PROOF

Suppose f is continuous at a. Let a_n be a sequence in X such that $a_n \to a$ as $n \to \infty$. If ε is a positive real number, let $V = \{x \in X \mid \rho(f(x), f(a)) < \varepsilon\}$.

Then V is an open neighborhood of a in X. Choose a positive integer P such that $a_n \in V$ for every $n \geq P$.

Given $\rho(f(a_n), f(a)) < \varepsilon$ for every $n \geq P$.

Therefore $f(a_n) \to f(a)$ as $n \to \infty$. This proves $1 \Rightarrow 2$

Conversely suppose f is not continuous at a. Choose $\eta > 0$ such that there exist no $\delta > 0$ satisfying for $x \in X$ $d(x,a) \Rightarrow \rho(f(x), f(a)) < \eta$.

For every positive integer n. Choose $w_n \in X$ such that $d\left(w_n, a\right) < \frac{1}{n}$ but

$\rho\left(f\left(w_n\right), f\left(a\right)\right) \geq \eta$ then $w_n \to a$ as $n \to \infty$ but $f\left(w_n\right)$ does not converge

to $f\left(a\right)$. This proves 2 is false if 1 is false . Hence $2 \Rightarrow 1$.

2.26 UNIFORM CONTINUITY

Let $\left(S, d\right)$ and $\left(Y, \rho\right)$ be metric spaces and $f : S \to Y$ a mapping. We say that f is uniformly continuous if to every positive real number ε, there corresponds a positive real number δ such that $\rho\left(f\left(x\right), f\left(v\right)\right) < \varepsilon$ for every pair $x, v \in S$ such that $d\left(x, v\right) < \delta$.

2.26.1 EXAMPLE

Let $\left(S, d\right)$ be a metric space

Given $a \in S$ define

$f : S \to R$ by $f\left(x\right) = d\left(a, x\right)$.Then f is uniformly continuous.

PROOF

For every pair $x, y \in S$

$f(x) = d(a,x) \leq d(a,y) + d(x,y) = f(y) + d(x,y)$ and

$f(y) = d(a,y) \leq d(a,x) + d(x,y) = f(x) + d(x,y)$

Hence $\left| f(x) - f(y) \right| \leq d(x,y)$

Given a positive real number ε, let $\delta = \varepsilon$

Then $\left| f(x) - f(y) \right| < \varepsilon$ for every pair $x, y \in S$ such that $d(x,y) < \delta$.

2.27 ORTHOGONAL AND ORTHONORMAL SETS

A set S in an inner product space E is called an *orthogonal* set if

$\langle x, y \rangle = 0$ for each $x, y \in S, x \neq y$.

The set S is called *orthonormal* if it is an orthogonal set and $\|x\| = 1$ for each $x \in S$.

2.27.1 THEOREM

Orthogonal systems are linearly independent.

PROOF

Let S be an orthogonal system. Suppose $\sum_{k=1}^{n} \alpha_k x_k = 0$ for some $x_1, ..., x_n$ and scalars $\alpha_1, ..., \alpha_n$. Then $0 = \sum_{m=1}^{n} \langle 0, \alpha_m x_m \rangle = \sum_{m=1}^{n} \left\langle \sum_{k=1}^{n} \alpha_k x_k, \alpha_m x_m \right\rangle$

$$= \sum_{m=1}^{n} |\alpha_m|^2 \|x_m\|$$

This implies that $\alpha_m = 0$ for each $m \in N$, where N is a positive integer. Thus $x_1, ..., x_n$ are linearly independent.

2.27.2 ORTHONORMAL SEQUENCE

A sequence of vectors which constitute an orthonormal system is called an *orthonormal sequence*

2.27.3 PROPERTIES OF ORTHONORMAL SYSTEMS

2.27.4 THEOREM (Pythagorean Formula)

If $x_1, \ldots\ldots, x_n$ are orthogonal vectors in an inner product space, then

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \sum_{k=1}^{n}\|x_k\|^2 .$$

PROOF

If $x_1 \perp x_2$, then $\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$.

Thus, the theorem is true for $n = 2$.

Assume now that the theorem hold for $n-1$, that is

$$\left\|\sum_{k=1}^{n-1} x_k\right\|^2 = \sum_{k=1}^{n-1}\|x_k\|^2$$

Set $x = \sum_{k=1}^{n-1} x_k$ and $y = x_n$

Since $x \perp y$, we have

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \|x + y\|^2$$

$$= \|x\|^2 + \|y\|^2$$

$$= \sum_{k=1}^{n-1}\|x_k\|^2 + \|x_n\|^2 = \sum_{k=1}^{n}\|x_k\|^2$$

This proves the theorem.

2.27.5 THEOREM (Bessel's Equality and Inequality)

Let $x_1, x_2, ...$ be an orthonormal sequence of vectors in an inner product space E.

Then for every $x \in E$, we have

$$\left\| x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\|^2 + \sum_{k=1}^{n} \left| \langle x, x_k \rangle \right|^2 \quad = \|x\|^2$$

It follows that $\sum_{k=1}^{n} \left| \langle x, x_k \rangle \right|^2 \leq \|x\|^2 \;\; \forall n \geq 1$

PROOF

$$\left\| x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\|^2$$

$$= \left\langle x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k , x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\rangle$$

$$= \|x\|^2 + \sum_{k=1}^{n} \left| \langle x, x_k \rangle \right|^2 - 2\sum_{k=1}^{n} \left| \langle x, x_k \rangle \right|^2$$

$$= \|x\|^2 - \sum_{k=1}^{n} \left| \langle x, x_k \rangle \right|^2$$

$$\Rightarrow \|x\|^2 = \left\| x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\|^2 + \sum_{k=1}^{n} \left| \langle x, x_k \rangle \right|^2$$

It follows that $\sum_{k=1}^{n} \left| \langle x, x_k \rangle \right|^2 \leq \|x\|^2 \;\; \forall n \geq 1$Let $n \to \infty$. Then $\sum_{k=1}^{\infty} \left| \langle x, x_k \rangle \right|^2 \leq \|x\|^2$

2.28 COMPLETE ORTHONORMAL SEQUENCE

An othonormal sequence (x_n) in an inner product space E is said to be *complete* if for every $x \in E$ we have

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$$

2.29 ORTHONORMAL BASIS

An orthonormal system B in an inner product space E is called an *orthonormal Basis* if every $x \in E$ has a unique representation

$$x = \sum_{n=1}^{\infty} \alpha_n x_n .$$

Where $\alpha_n \in C$, where C is a complex number and $x_n 's$ are distinct elements of B.

2.29.1 THEOREM

An orthonormal sequence (x_n) in a Hilbert space H is complete if and only if $\langle x, x_n \rangle = 0$ for all $n \in N$ implies $x = 0$

PROOF

Suppose (x_n) is complete orthonormal sequence in H. Then every $x \in H$

has the representation

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$$

Thus, if $\langle x, x_n \rangle = 0 \ for\ every\ n \in N$, then $x = 0$

Conversely, suppose $\langle x, x_n \rangle = 0 \ for\ all\ n \in N$ implies $x = 0$.

Let x be an element of H. Define

$$y = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$$

Since, for every $n \in N$,

$$\langle x - y, x_n \rangle = \langle x, x_n \rangle - \left\langle \sum_{k=1}^{\infty} \langle x, x_k \rangle x_k, x_n \right\rangle$$

$$= \langle x, x_n \rangle - \sum_{k=1}^{\infty} \langle x, x_k \rangle \langle x_k, x_n \rangle$$

$$= \langle x, x_n \rangle - \langle x, x_n \rangle = 0$$

We have $x - y = 0$ and hence $x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$

2.29.2 THEOREM (Parseval's Formula)

An orthonormal sequence (x_n) in a Hilbert space H is complete if and only if

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \quad for\, every \quad x \in H \text{ .}$$

PROOF

Let $x \in H$. By the Bessel equality, for every $n \in N$, we have

$$\left\| x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x, x_k \rangle|^2 \qquad (3)$$

If (x_n) is a complete sequence, then the expression on the let in (3)

converges to zero as $n \to \infty$. Hence

$$\lim_{n \to \infty} \left[\|x\|^2 - \sum_{k=1}^{n} |\langle x, x_k \rangle|^2 \right] = 0$$

Therefore the theorem holds.

Conversely, if the theorem holds, then the expression on the right in (1)

converges to zero as $n \to \infty$, and thus

$$\lim_{n \to \infty} \left\| x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\|^2 = 0$$

This proves that (x_n) is a complete sequence.

2.30 ORTHOGONAL COMPLEMENT

Let S be a non-empty subset of a Hilbert space, H. An element $x \in H$ is said to be orthogonal to S, denoted by $x \perp S$, if $\langle x, y \rangle = 0 \; \mathit{for\,every}\; y \in S$.

The set of all elements of H orthogonal to S, denoted by $S^{\perp}$, is called the *orthonormal complement* of S. That is

$$S^{\perp} = \{x \in H, x \perp S\}$$

If $x \perp y$, for every $y \in H$, then $x = 0$. Thus, $H^{\perp} = \{0\}$. Similarly,

$$\{0\}^{\perp} = H$$

Two subsets A and B of a Hilbert space are said to be orthogonal if $x \perp y$ for every $x \in A$ and $y \in B$. This is denoted by $A \perp B$..

2.30.1 THEOREM

For any subset S of a Hilbert space H, the set $S^{\perp}$ is a closed subspace of H

PROOF

If $\alpha, \beta \in C$ and $x, y \in S^{\perp}$, then

$$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle = 0 \; \mathit{for\,every}\; z \in S.$$

Thus $S^{\perp}$ is a vector subspace of H.

We next prove that $S^{\perp}$ is closed.

Let $(x_n) \in S^{\perp}$ and $x_n \to x$ for some $x \in H$.

From the continuity of the inner product, we have

$$\langle x, y \rangle = \left\langle \lim_{n\to\infty} x_n, y \right\rangle = \lim_{n\to\infty} \langle x_n, y \rangle = 0 \quad for\, every\; y \in S .$$

This shows that $x \in S^{\perp}$ is closed.

2.30.2 THEOREM(Projection theorem)

Let H be a Hilbert space and M a closed subspace of H . For arbitrary vector x in H , there exist a unique vector $m^* \in M$, such that $\|x - m^*\| < \|x - m\|$ for all $m \in M$.

Furthermore, $m^* \in M$ is the unique vector if and only if $\left(x - m^*\right) \perp M$

2.31 DIRECT SUM

Let V be a vector space. V is said to be the $direct\, sum$ of two subspaces M and N of V written $V = M \oplus N$ if each $x \in V$ can be represented uniquely as

$x = m + n$ where $m \in M$ and $n \in N$.

2.31.1 THEOREM(direct sum)

Let M be a closed subspace of a Hilbert space, H .Then $H = M \oplus M^{\perp}$

PROOF

Let $x \in H$ be arbitrary. By the projection theorem, there exist a unique vector

$m^* \in M$ such that $\left\| x - m^* \right\| \le \left\| x - m \right\|$ for all $m \in M$ and let

$n^* := x - m^* \in M^{\perp}$.consequently, we can write

$x = m^* + \left(x - m^* \right) := m^* + n^*$ with $m_1 - m^* + n_1 - n^* = 0$ $m^* \in M$ and
$n^* \in M^{\perp}$.

It remains now to show that this representation is unique.

Suppose that $x = m_1 + n_1$ with $m_1 \in M$ and $n_1 \in M^{\perp}$ is another

representation of x.

Then $m^* + n^* = m_1 + n_1$ so that $m_1 - m^* + n_1 - n^* = 0$. But $\left(m_1 - m^* \right)$ and
$\left(n_1 - n^* \right)$

are orthogonal. Hence by Pythagoras theorem

$$\left\| \left(m_1 - m^* \right) + \left(n_1 - n^* \right) \right\|^2 = \left\| m_1 - m^* \right\|^2 + \left\| n_1 - n^* \right\|^2 = 0$$

This implies, $\left\| m_1 - m^* \right\| = 0$ and $\left\| n_1 - n^* \right\| = 0$. Hence $m_1 = m^*$ and $n_1 = n^*$,

establishing the uniqueness of the representation. Thus $H = M \oplus M^{\perp}$

2.31.2 THEOREM(Riesz Representation Theorem)

This theorem shows that any bounded linear functional on a Hilbert space can be represented as an inner product with a unique vector in H

Let H be a Hilbert space and let f be a bounded linear functional on H. Then,

(i) There exist a unique vector $y_0 \in H$ such that

$f(x) = \langle x, y_0 \rangle$ for each $x \in H$ (1)

(ii) Moreover, $\|f\| = \|y_0\|$

PROOF

(i) Let $M = \{x \in H : f(x) = 0\}$.

Then clearly M is a closed subspace of H. By the direct sum theorem,

$H = M \oplus M^{\perp}$ and $M \cap M^{\perp} = \{0\}$.

If $M = H$, we are done because we can take the unique vector $y_0 \in H$ as $y_0 \equiv 0$ and (1) is satisfied.

If $M \neq H$, let $z \in M^{\perp}$. Then $f(z) \neq 0$

Let $x \in H$ be arbitrary and let

$$u = x - \frac{f(x)}{f(z)} z \qquad (2)$$

Applying f to (2) we have

$f(u) = f(x) - f(x) = 0$.

Which implies $u \in M$.

Thus, $u \perp z$, that is $\langle u, z \rangle = 0$.

Taking inner product of (2) with z we get

$\left\langle u, z \right\rangle = \left\langle x - \frac{f(x)}{f(z)} z, z \right\rangle = \langle x, z \rangle - \frac{f(x)}{f(z)} \langle z, z \rangle = 0$ and so,

$$f(x) = \frac{f(z)}{\langle z, z \rangle} \langle x, z \rangle$$
$$= \left\langle x, \left(\overline{f(z)} / \overline{\langle z, z \rangle} \right) z \right\rangle$$
$$= \langle x, y_0 \rangle$$

and so $y_0 = \left(\overline{f(z)} / \overline{\langle z, z \rangle} \right) z$

For arbitrary $z \in M^{\perp}$

(ii) From (1), $|f(x)| \leq \|x\| . \|y_0\|$ and so $\|f\| \leq \|y_0\|$.

Also

$$f(y_0) = \langle y_0, y_0 \rangle = \|y_0\|^2$$

So that $\|f\| = \|y_0\|$

For uniqueness, let $y_1, y_2 \in H$ satisfy $f(x) = \langle x, y_1 \rangle = \langle x, y_2 \rangle$ for every $x \in H$.

Then,

$\langle x, y_1 - y_2 \rangle = 0$ for all $x \in H$.

In particular, $\langle y_1 - y_2, y_1 - y_2 \rangle = 0$ so that

$\|y_1 - y_2\|^2 = 0$ Which yields $y_1 = y_2$

2.32 SEPARABLE HILBERT SPACES

A Hilbert space is called *separable* if it contains a complete orthonormal sequence. Finite dimensional Hilbert spaces are separable.

2.33 ISOMORPHISM

A Hilbert space H_1 is said to be *isomorphic* to a Hilbert space H_2, if there exist a one-to-one linear mapping T from H_1 onto H_2 such that

$$\langle T(x), T(y) \rangle = \langle x, y \rangle \qquad *$$

For every $x, y \in H_1$.

Note that * implies $\|T\| = 1$ because $\|T(x)\| = \|x\|$ for every $x \in H_1$

2.33.1 THEOREM

Let H be a separable Hilbert space. If H is infinite dimensional, then it is isomorphic to l_2.

PROOF

Let (x_n) be complete orthonormal sequence in H. If H is infinite

dimensional, then (x_n) is an infinite sequence. Let x be an element of H.

Define $T(x) = (\alpha_n)$, where $\alpha_n = \langle x, x_n \rangle$, $n = 1, 2, \ldots$

T is one-to-one mapping from H to l_2. It is clearly a linear mapping.

Moreover, for $\alpha_n = \langle x, x_n \rangle$ and $\beta_n = \langle y, y_n \rangle$, $x, y \in H, n \in N$, where N is a positive integer.

We have $\langle T(x), T(y) \rangle = \langle (\alpha_n), (\beta_n) \rangle$

$$= \sum_{n=1}^{\infty} \alpha_n \overline{\beta_n}$$

$$= \sum_{n=1}^{\infty} \langle x, x_n \rangle \overline{\langle y, x_n \rangle}$$

$$= \sum_{n=1}^{\infty} \langle x, \langle y, x_n \rangle x_n \rangle = \left\langle x, \sum_{n=1}^{\infty} \langle y, x_n \rangle x_n \right\rangle = \langle x, y \rangle$$

Thus T is isomorphism from H onto l_2.

CHAPTER THREE

In this chapter we discuss two topologies on the dual of a Banach space.

3.1 DEFINITION

Let V be a Banach space over R. For every bounded linear functional $f \ onV$,

there is a constant positive real number λ such that $\left\| f\left(x\right)\right\| \le \lambda \left\| x\right\|$. Hence

$$\frac{\left\| f\left(x\right)\right\|}{\left\| x\right\|} \le \lambda \ for\, all\; x \in V \; such\, that\; x \ne 0$$

We define the norm, $\left\| f\right\|_*$ as the least upper bound of all set of the form

$$\frac{\left\| f\left(x\right)\right\|}{\left\| x\right\|} \; such\ that\; x \in V \; and\; x \ne 0 .$$

This defines a norm, $\left\|,\right\|_* \ onV^*$. Indeed $\left(V^*,\left\|,\right\|_*\right)$ is also a normed vector space.

Recall that in section **2.12** , the topology induced by $\left\|,\right\|_*$ was discussed. Also in section **3.3** another topology, the weak topology will be described.

As mentioned earlier, in an infinite dimensional Hilbert we give an example of a closed bounded subset which is not compact.

3.2 EXAMPLE

Let $S = \{x \in l_2 \mid \|x\| \leq 1\}$

Then S is a closed and bounded subset in the Hilbert space l_2, but S is not compact.

PROOF

In l_2 define a sequence e_n as follows: $e_n(k) = \begin{cases} 0 \ if \ k \neq n \\ 1 \ if \ k = n \end{cases}$

Then $\{e_n \mid n = 1, 2, 3, \ldots.\}$ is an orthonormal set in l_2.

Assume that S is compact in the normed vector space $\left(V^*, \|,\|_*\right)$. Then the sequence e_n has a convergent subsequence e_{n_k}. Choose a positive integer q such that

$\left\|e_{n_j} - e_{n_k}\right\| < 1$ whenever $j \geq q$ and $k \geq q.$

Then there is a contradiction $\sqrt{2} = \left\|e_{n_q} - e_{n_{q+1}}\right\| < 1$.

The assumption is false. Hence S is not compact.

Finally we look for a topology on l_2 for which S is compact.

3.3 DEFINITION

There is the weak topology of V^* which is the topology of V^* generated by all set of the form

$U(x,G)=\{f \in V^* \mid f(x)\in G\}$ for each $x \in V$, and each open set

$G \subset R$. Let the sets $U(x,G)$ be a subbasis for the topology in V^*. This Topology is called the weak topology in V^* determined by V.

3.3.1 EXAMPLE

Let $G=\left\{t \in R \mid |t| < \dfrac{1}{2}\right\}$

Given $a \in V$, we find that $U(a,G)=\left\{f \in V^* \mid |f(a)| < \dfrac{1}{2}\right\}$.

3.4 THEOREM

If V is a Banach space and the dual V^* is given its weak topology then, the closed unit ball $S \subset V^*$ is compact.

PROOF

For each $x \in V$. let R_x be the set of real numbers. The space $\prod[R_x \mid x \in V]$ evidently contains V^*, and the weak topology in V^* is the one induced by the topology in $\prod[R_x \mid x \in V]$. Let K be the subset of $\prod[R_x \mid x \in V]$ of points whose x-coordinate has absolute value not greater than $\|x\|$, for every $x \in V$.

By the theorem (that is if $X_\alpha, \alpha \in A,$ are compact spaces, then
$X = \prod[X_\alpha | \alpha \in A]$ is

compact.), K is compact. We show that the closed unit ball $S \subset V^*$ is a weakly

closed subset of K.

Let $\psi \in \overline{S}$, the weak closure of S in K. Then $\psi \in K.$ We show that ψ is linear.

For this, let $x, y, x+y \in V.$ let $\in > 0.$ By the definition of the topology

$\prod[R_x | x \in V]$, there is an $x^* \in V^*$ such that

$|x^*(x) - \psi(x)| < \varepsilon, |x^*(y) - \psi(y)| < \varepsilon,$
and $|x^*(x+y) - \psi(x+y)| < \varepsilon.$

Since $x^*(x+y) = x^*(x) + x^*(y),$ it follows that

$|\psi(x+y) - \psi(x) - \psi(y)| < 3\varepsilon.$

Since this holds for every $\varepsilon > 0,$

$\psi(x+y) = \psi(x) + \psi(y).$ It is just as easy to show that for every $x \in V$ and

every $a \in R, \psi(ax) = a\psi(x).$ Since $|\psi(x)| \le \|x\|,$ for every $x \in V,$ it follows that

$\psi \in B.$ Hence S is closed and we have proof theorem **3.4.**

CONCLUSION

The conclusion is that when l_2^* is given its normed topology then l_2^* is isomorphic to l_2 .In this case $\left\{x \in l_2^* \mid \|x\| \leq 1\right\}$ is not compact. On the other hand l_2^* can be given the weak topology. In this case $\left\{x \in l_2^* \mid \|x\| \leq 1\right\}$ is compact.

It must be emphasized that the normed topology is the natural topology of V^*.

There are times when the weak topology is needed.

RECOMMENDATION

On the same vector space several unequal topologies may be defined and when you want to choose one you have to decide your purpose.

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