CONSTRUCTING OPTIMAL STOCK PORTFOLIO WITH MARKOWITZ MODEL

By

SAMUEL DARKO

(B.A. Hons)

A Thesis submitted to the Department of Mathematics (Institute of Distance Learning), Kwame Nkrumah University of Science and Technology in partial fulfilment of the requirements for the degree of

MASTER OF SCIENCE

(Industrial Mathematics)

College of Science

May, 2012

CONSTRUCTING OPTIMAL STOCK PORTFOLIO WITH MARKOWITZ MODEL

BY

SAMUEL DARKO (B.A HONS.)

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS (INSTITUTE OF DISTANCE LEARNING) OF THE KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE, COLLEGE OF SCIENCE

MAY, 2012.

DECLARATION

I declare that this thesis submitted to the Graduate School of Kwame Nkrumah University of Science and Technology (Institute of Distance Learning) is my own work towards the Msc degree and that, except for references to other researchers work which have duly acknowledged, this thesis has not been submitted to any other university for the award of a degree.

Samuel F. Darko (PG4065810)	…………….	…………….
(Name of Student & ID)	Signature	Date

Certified by:

Dr. Lord Mensah	…………….	..…………
(Name of Supervisor)	Signature	Date

Certified by:

Mr. F. K. Darkwah	…………….	…………
Head of Department	Signature	Date

ABSTRACT

A portfolio is a collection of financial assets consisting of investment tools such as stocks, bonds, gold, foreign exchange, asset-backed securities, real estate certificates and bank deposit which are held by a person or a group of persons. In Ghana, constructing portfolio with standardized optimization model still remains a myth. In this paper, we analyze the applicability of the classical Markowitz model on the Ghana Stock Exchange. We further determine which Stock-index is profitable for an investor; thus should the investor invest in the GSE-All-Share Index, Non-financial Index, or the Financial Index given the current performance of the indices. Historical monthly data of stock prices, market capitalization and dividend per share from 2007 to 2010 were used to compute the market indices.

The study concludes that it is profitable for an investor to invest 83.44% of his capital in the non-financial index while investing 16.56% in the financial index. We further conclude that the Ghanaian stock market obeys the tenet of the Markowitz model.

ACKNOWLEGEMENT

I wish to take this opportunity to acknowledge Dr Lord Mensah, my supervisor for his immense contribution for the success of this work. I also appreciate all the Lecturers of the Mathematics Department for their varied assistances and motivation.

DEDICATION

I dedicate this paper to God Almighty through His son Jesus Christ for seeing me through this work, and my mother Akosua Adomah who has assisted me throughout my education to this level.

LIST OF FIGURES. **PAGES**

LIST OF ABBREVIATIONS.

MV…………………………………………………………………………………………………Mean Variance

SAA……………………………………………………………………………………………… Strategic Asset Allocation.

MLPM……………………………………………………………………………………………Mean-Lower Partial Moment

CAPM………………………………… ……………………………………………………………Capital Asset Pricing Model

APT…………………………………… ……………………………………………………………..Arbitrage Price Theory

MADM……………………………….. ……………………………………………………………Mean Absolute Deviation Model

S&P 500………………………………. ……………………………………………………………..Standard & Poor 500 Index

LAM………………………………………………………………………………………………Limited Asset Markowitz Model

GSE………………………………………………………………………………………………Ghana Stock Exchange.

TABLE OF CONTENTS

Contents

CHAPTER 1

1.0 INTRODUCTION.

1.1 Background of Study.

A portfolio is a collection of financial assets consisting of investment tools such as stocks, bonds, gold, foreign exchange, asset-backed securities, real estate certificates and bank deposit which are held by a person or a group of persons. Portfolio construction is a fundamental problem in financial economics, and plays a significant role in both theory and practice. The pioneering work of portfolio construction was done by Harry M. Markowitz in 1952 in his article "portfolio selection", where a quantitative approach for portfolio selection was first presented. He developed mathematical framework for the problem and obtained a feasible solution to the problem which was simple and intuitively appealing. He considered a single-period economy and formulated the portfolio selection problem as a static mean-variance optimization problem, where the variance or the standard deviation was used as a measure of risk and mean as a measure of portfolio return. The simplified framework of the Markowitz model is justified when the distribution of return is normal, or when the investor has a quadratic utility function. In the Markowitz mean-variance portfolio selection, the optimal portfolio selection is done by minimizing the variance of the portfolio's return for a given level of expected portfolio return, or maximizing the expected portfolio return for a given level of variance of the portfolio return. The mean-variance paradigm also provides a simple geometric representation for portfolio selection including investment opportunities, portfolio diversification and efficient frontier. The basic motive behind portfolio construction is risk dispersion. Since the returns on the assets constituting portfolio do not move in the same direction, the risk of the portfolio will be lower than that of the single asset. From this principle it follows that the traditional portfolio management approach is based on the rule of increasing the number of assets in a portfolio. This approach could be described as "not to put all eggs in one basket" (Fisher, Jordan, 1991).Markowitz again states that, the portfolio's return (mean) and its variance are the whole criteria for portfolio selection and construction. These parameters can be used as a possible maxim for how investor ought to select his/her portfolio. It will interest you to know that, the whole model is based on an economic fact of "Expected Utility". The concept of utility here is based on the fact that different investors have different investment goals and can be satisfied in different ways. Consequently; every investor seeks to maximize their utility (satisfaction) by maximizing expected return and minimizing risk.

Prior to Markowitz paper in 1952, Hicks mentioned the necessity of improvement on the theory of money in 1935.He introduced risk in his analysis and stated that "the risk -factor comes into our problem in two ways: First, as affecting the expected period of investment, and second as affecting the expected net yield of investment. On his work William Sharpe (1964) and Litner (1965b) almost simultaneously developed a model to price capital asset, famously called Capital Asset Pricing Model (CAPM). This model relates expected return to a measure of risk that incorporate what some consider to be the "only free launch in finance economics"; Diversification. This measure now known as beta, use the theoretical result that, diversification allows investors to escape company's specific risk. The Markowitz model could be summarized as follows; one needs to:

• Calculate the expected return rates for each stock to be included in the portfolio,

• Calculate the variance or standard deviation (risk) for each stock to be included in the Portfolio,

• Calculate the covariance or correlation coefficients for all stocks, treating them as pairs.

Later studies by Sharpe (1964), Lintner (1965),Mossin (1966) and Zorlu (2003) on portfolio construction further investigated the trend of prices in case all savers invest in financial assets and particularly in share certificates in accordance with the modern portfolio theory.

Even though, it is no secret that the Markowitz mean-variance model has empirical set backs, it is nevertheless the most widely used model in both academic and real-world applications (Fama, 2004). In this paper we shall immensely discuss the Markowitz model in finance theory as applied on the Ghana stock exchange. The Markowitz optimization problem can be summarized as: Maximizing the expected return subject to the variance of return, Minimizing the variance for a lower limit on the expected return, and Maximizing the Sharpe ratio.

The above optimization problem could be solved by the use of the Lagrange multiplier which yields the maximum or the minimum solution to the problem; but in this paper the optimization problem is solved with the excel solver.

1.2 Problem statement

Most investors and portfolio managers in Ghana seek to optimally construct their stock portfolio on the Ghana stock exchange in order to satisfy their investment goals. However the problem invariably remains “which combination of sets of portfolio must he select for him to reap maximum return given a level of risk? Or conversely, which sets of portfolio would yield a minimum risk given a level of return?” Must he select stocks from the Financial, the Non-financial index or the entire market index in order for him to reap the expected higher return?

1.3 Objectives of Study.

The specific objectives of this paper are as follows:

a) To establish the applicability of the Markowitz model on the Ghana Stock Exchange.
b) To inform investors that diversification reduces unsystematic risk of portfolio.
c) Analysis is required for portfolio construction because of the infinite number of portfolio of risky assets.

1.4 Methodology.

In our attempt to resolve the portfolio optimization puzzle in the light of Markowitz model, we shall make use of some statistical parameters, namely; mean, variance (or standard deviation), covariance and correlation matrix for the Markowitz model formulation. We apply the Markowitz model to solve real world problem based on monthly returns of stocks listed on the Ghana stock Exchange. Data spanning a four year period are collected from the Ghana stock exchange to enable us formulates our optimization model. We shall use excel solver to run the optimization problem to arrive at our optimal solution. Other information and data may be acquired from some journals on finance, and internet references and the University library.

1.5 Justifications

At the end of this paper, readers and users of this paper would be able to familiarize themselves on portfolio selection strategies on Ghana stock exchange. Also this work will demystify the naive beliefs on equity management and sharpen the investment know how of the Ghanaian populace, thereby attracting more people into the investment bracket for general economic development.

1.6 Structure of thesis.

The thesis is divided in five different chapters, each covering one topic. Chapter two presents a general literature review of the basic theory of the Markowitz model and its current state of development in some stock markets around the world. Chapter three discusses the mathematical and statistical estimates of the key input variables for portfolio theory, and also the formulation of the Markowitz Model. Chapter four opens with the data collection and analysis of returns on shares traded on the Ghana stock exchange. The final chapter, five, highlights conclusions and recommendations drawn from the use of the Markowitz model in the light of its limitations and suggest area of further research.

CHAPTER 2

2.0 LITERATURE REVIEW

Markowitz's ground-breaking research on portfolio optimization in March 1952 in an article titled, "portfolio selection" in the journal of finance afforded him to be called the father of modern portfolio theory. He was accordingly awarded the Nobel Prize for Economics in 1990 together with William Sharpe and Merton Miller. Prior to Markowitz work, investors focused on assessing the risk and return of individual securities in constructing their portfolios. Standard investment policy was to identify these securities that offered the best opportunities for gain with the least risk and then construct a portfolio from these securities. Following this advice, an investor might conclude that, bank stocks offer good risk- return characteristics, and therefore compile a portfolio entirely from them. Intuitively, this will be inappropriate. Markowitz formalized this intuition by suggesting that, the value of a security to an investor is best evaluated by its mean, standard deviation/variance, and its correlation to other securities in the portfolio. This audacious suggestion by Markowitz amounted to ignoring a lot of information about the firm (its earning, dividend policy, capital structure, market and competitor) and calculating a few simple statistics. He proposed that investors should focus on selecting portfolio based on their overall risk-return characteristics instead of merely compiling portfolios from securities that each individual has attractive risk-return characteristic. In a nutshell investors should select portfolio not individual securities.

He determined that one of the principal objectives of investors, besides the maximization of the returns of their portfolio is to diversify away as much risk as possible. He maintained that investors select assets in such a way that the risk of their portfolio matches with their risk preferences. In other words, he suggested that, individuals who can not bear risk will invest in asset with low risk, whereas people more comfortable with risk will accept investments of higher risk. His work also suggests that, the trade-off between risk and return is different for each investor. He derived the 'critical line algorithm' which identifies all feasible portfolios from a given set of assets that minimizes risk for a given return, and maximizes return for a given level of risk which is known as efficient frontier. To derive the efficient frontier requires three variables (Markowitz 2000, p.4) namely; (a) the expected return of the asset. (b) the expected variance of the asset and (c) the cross-correlation between the asset classes. Initially, the process of deriving the critical line involved solving for corner portfolio along the line. These corner portfolios included the maximum return portfolio, the minimum variance portfolio, and any number of portfolios in between. Computing power technology is now able to derive the magnitude of portfolios that make up the critical line, otherwise known as the efficient frontier. An investor who can tolerate more risk might choose a portfolio on the higher point of the frontier, while a more risk averse investor would be more likely to choose a portfolio at the lowest point on the frontier. A portfolio way below the efficient frontier is thus inefficient, and hence would require an adjustment to the asset allocation in order for the investor to move close or on the curve, known as strategic asset allocation (Statman, 2001, p.133). Brennan, Schwartz and Lagnado(1997) coined the term "Strategic Asset Allocation" (SAA) to designate optimal asset allocation rebalancing strategies in the face of changing investment opportunities. SAA portfolios are a combination of two portfolios. The first one is a short-term mean variance efficient portfolio. It reflects short-term or myopic considerations, while the second portfolio which Merton (1969, 1971, 1973) called "inter-temporal hedging portfolio" reflects long-term dynamic hedging consideration.

We can construct a large number of portfolios by combining securities and by varying proportions of investment among assets. Among the portfolio formed, some are efficient, while many others are inefficient. The set of portfolios that maximize expected return for varying level of risk or minimize risk for a varying level of expected return is known as efficient set. The investor will choose portfolio from these efficient portfolios. The optimal-risk portfolio is usually determined to be somewhere in the middle of the curve, because as one goes higher up the curve he/she takes on proportionately more risk for a lower investment return. But low risk/ low return portfolio are pointless (when one moves down the curve), because he/she can achieve a similar return by investing in risk-free return assets like government securities.

Markowitz formulated the portfolio problem as a choice of the mean and variance of a portfolio of assets. He proved the fundamental theorem of mean-variance portfolio theory; namely holding constant variance, maximize expected return, and holding constant expected return and minimized variance. Markowitz developed the theory of portfolio choice in an uncertain future. He quantified the difference between the risk of portfolio assets taken individually and the overall risk of the portfolio. He demonstrated that the portfolio risk came from the co-variances of the asset that made up the portfolio. The marginal contribution of a security to the portfolio return variance is therefore measured by the co-variance between the security's return and the portfolio return, but not by the variance of the security itself. The total risk of a portfolio could be decomposed into systematic risk (also known as the market risk, which can not be eliminated, for example, interest rate, wage levels, inflation rate, and foreign exchange) and unsystematic risk (which could be eliminated through diversification) (Statman, 1987).

Although, it is generally true that, when stocks are selected randomly and combined in equal proportions into a portfolio Ferri (2002,p.186), the total risk declines as indicated above, Evans and Archer (1968) observed that the risk reduction effect diminishes rapidly as the number of stocks increase. They observed that the economic benefits of diversification are exhausted when a portfolio contains ten or more stocks. Evans and Archers conclusion has been cited in many text books. For example, Francis (1986) wrote; "portfolio managers should not be overzealous and spread their assets over too many assets. The maximum benefit of diversification is achieved if 10 or 15 different assets are selected for the portfolio. Further spreading of the portfolio's assets is superfluous diversification".

The Markowitz model has the following assumptions: (1) that an investor is concern with return distribution over a single period. (2) Investors seek to maximize the expected return of total wealth. (3) all investors are risk-averse, i.e they will only accept a higher risk if they are compensated for higher expected return. (4) Investors based their investment decisions on the expected return and risk. (5) All markets are perfectly efficient. By a single period we mean that, investors make their portfolio decisions at the beginning of a period and then wait until the end of the period when the rate of return on their portfolio is realized. Also the investor can not make any intermediate changes in the composition of his portfolio; and finally the investor makes his decision with the objective of maximizing expected utility of wealth at the end of the period (final wealth).

The Markowitz approach is often describe as a mean-variance approach because, it only takes those two parameters, mean return and variance of return into account(i.e the first two moments of their distribution) to characterize the investors portfolio. The expected return of the portfolio is measured by the mean return, while the risk of the portfolio is measured by the variance. The variance facilitates simple modeling, and also is a good measure of risk under the assumption that returns are normally distributed. The theory developed by Markowitz is also based on maximizing the expected utility of the investor's terminal wealth. This utility function is defined according to the expected return and the standard deviation of the wealth. A number of studies have empirically investigated the ability of the mean-variance analysis to maximize the expected utility of an investor. Although the conclusion of these studies have been mixed, the general difficulties found have been accuracy in measuring the quality of mean-variance efficient solution and how well relying on only mean and variance would work in real asset allocation problem. Levy and Markowitz (1979) estimated the expected utility by a function of mean and variance of return of 149 mutual funds, and found that ordering a portfolio by mean variance rule was almost identical to the order obtained by using expected utility.

Pulley (1981) indicated that the mean variance formulation provides a very good local approximation to expected utility for more general utility functions using both monthly and semi-annual return data. The paper also suggested that the mutual fund manager should select portfolios which maximize utility for a wide class of individual investors having different utility functions and wealth levels regardless of the actual form of their utility functions.

Kroll *et al* (1984) also reported that the best mean-variance efficient portfolio has almost maximum obtainable expected utility, and the same is true even when 50% borrowing is allowed. Their work and that of Pulley (1981) both compared expected utility of mean-variance efficient portfolios to the expected utility of the optimal portfolio, but Kroll et al (1984) used annual holding period to pose a greater challenge for mean-variance approximation, since it was agreed by these researchers that the higher the portfolios variance, the less likely is the mean-variance approximation to do actual expected utility maximization.

However, the mean-variance model of Markowitz has received serious criticism. For example, Borch (1969) and Feldstein (1969) indicated that, the mean variance framework only leads to optimal decisions if utility functions are quadratic or investment returns are jointly elliptically (spherically) distributed. Therefore, Bawa and Luenberger (1977) proposed a portfolio model known as the Mean-Lower Partial Moments (MLPM) portfolio framework based on the concept of downside risk. This approach gained much popularity among investors in 1990s and seemed to have had superiority to the mean-variance framework (Grootveld and Hallerbach, 1999).

However, Grootveld and Hallerbach (1999) examined the differences and similarities between variance and downside risk measures, and published an article which demonstrated that, few members of the large family of downside risk measures possess better theoretical properties within a return-risk framework than does variance.

Moreover, the implementation of mean-downside risk portfolio model is much more tedious since there are no shortcuts in computing portfolio risk (Grootveld and Hallerbach, 1999). Consequently, the mean-variance model has remained the most robust portfolio framework in the recent years. Numerous researchers such as Huang and Litzenberger (1988), Elton and Gruber (1995), Elliot and Kopp (1999), Jorion (2003), Mercurio and Torricelli (2003), Prakash *et al* (2003), Ehrgolt *et al* (2004) Ambachtsheer (2005), Campell and Viceira (2005), Aquino (2006) and Ulucan (2007) Biggs and Kane (2009) have successfully continued to study and revised the mean-variance model.

The research by Ulucan (2007) investigated the optimal holding period (investment horizon) for the classical mean-variance portfolio model. He used the historical transaction record of Istanbul Stock Exchange ISE-100 index stock, and Athens Stock Exchange FTSE-40 index stocks data for empirical analysis. The results of the study showed that portfolio returns with varying holding period had a convex structure with an optimal holding period.

Hiroshi and Hiroaki (1991) demonstrated that portfolio optimization model using the mean-variance absolute deviation risk function could remove most of the difficulties associated with the classical Markowitz model, while maintaining its advantages over equilibrium models like CAPM,APT etc. In particular the absolute deviation risk model leads to a linear instead of a quadratic program, so that a large scale optimization problem consisting of more than 1000 stocks may be solved on a real time basis. Numerical experiments using the historical data of NIKKE 1225 stocks showed that the model generates a portfolio quite similar to that of the Markowitz model within a fraction of time required to solve the classical Markowitz approach.

Biggs and Kane (2009) dealt with the issue of buy-in thresholds in portfolio optimization using the Markowitz model. Their study suggests that optimal values of invested fraction calculation using for example, the classical minimum-risk problem can be unsatisfactory in practice, because they lead to unrealistically small holding of certain assets. They therefore introduced discrete restrictions on each invested fraction, and used a combination of local and global optimizations to determine satisfactory solutions.

Paudel (2006) investigated the applications of the Markowitz and Sharpe models in the Nepalese Stock Exchange. His aim for the study was to test whether both models of portfolio selection offer any better investment alternatives to the Nepalese investors. With a sample of 30 stocks traded on the Nepalese stock market, the study finds that, the application of these models offer better options for making decision in the choice of optimal portfolios in the Nepalese market.

Yang and Hung (2010) propose a generalized Markowitz portfolio investment model via adding measures of skewness and peakness into the original Markowitz investment model. With these third and fourth moments (i.e skewness and peakness) in the objective function, they found that the magnitude of risk and shapes of the efficient frontier differ from that of the classical model of Markowitz; and hence the original work of Markowitz can be seen as special case of the generalized model.

Xia Lau Yang (2006) made use of the Genetic Algorithm along with a dynamic portfolio optimized system to improve the efficiency of the stock portfolio. In addition to Genetic Algorithm and Mean-Variance models, he proposed a third method called Bayesian perspective. The research findings showed that the genetic algorithm is of higher return compared to the other two methods and simultaneously with less risk. Besides, the analysis proved that the selected portfolio bases on both models of genetic algorithm in comparison to those of mean variance and Bayesian methods are of less fluctuation.

Lin and Liu (2008) also modeled the Markowitz approach in three ways, considering the limitations of the least amount purchased. They indicated that, the genetic algorithm only gain close point to optimization in little time for those models. Aranda and Iba (2009) introduced a tree algorithm that was used for the optimization of the stock portfolio. The smaller stock portfolios were obtained here.

Plessis and Ward (2009) endeavored to apply the Markowitz theory to the Johannesburg Security Exchange (JSE) to establish whether an optimal portfolio can be identified and used as an effective trading rule. In their work, weekly data covering 11 years on the top 40 JSE companies was analyzed to construct Markowitz mean variance optimized portfolios using ex-ante data. The optimal portfolio was then selected and rebalanced periodically, and the returns compared against JSE ALSI 40 index. The study found that the trading strategy significantly outperformed the market in the period under review.

Mwambi and Mwamba (2010) also investigated an alternative investment strategy to portfolio optimization model in the framework of the mean variance portfolio selection model. To differentiate it from the ubiquitously applied mean variance model of Markowitz, which is constructed on the assumption that returns are normally distributed, their model makes two assumptions; namely, that asset prices follow a geometric Brownian motion, and also assets prices are log-normally distributed (i.e continuously compounded returns are normally distributed). The model was then applied to five randomly selected stocks from JSE and compared to the Markowitz model. It was observed that while the Markowitz model is static one period strategy (buy and hold) and has a fixed time horizon, the log-normal strategy was dynamic and can be applied to any rebalancing period such as a year, month, week or a day. They however opined that the classical Markowitz approach was still relevant to the JSE.

Maharakkhaka (2011) evaluated the performance of the mean variance efficient approximation to maximize expected utility. By assuming that there are three classes of asset in the portfolio, namely; Security Exchange of Thailand (SET) Index, Thai investment grade corporate bond Index, and Thai government Treasury bill. He used monthly returns of these assets to compare maximum expected utility of the mean variance efficient portfolio to maximum expected utility derived from direct optimization. The findings indicate that, though picking the portfolio on the basis of the mean variance criteria does not lead to maximum expected utility, but the mean variance model is still relevant to Thailand Security Market. The performance of the mean variance approximation shown in the study was not much different from selecting naïve portfolio where investors easily put equal proportion of investment on each asset in their portfolio. Additionally, investors with various utility functions are found to require significant optimization premium to bring up their welfare to the level achieved by holding expected utility maximization portfolio.

Bai, Liu and Wong (2007) demonstrated that, the so called departure of the mean variance optimization model from its theoretical value is a natural phenomenon and the estimated optimal return is always larger than its theoretical parameter. Thereafter, they developed a new bootstrap estimator for the optimal return and its asset allocation, and proved that these bootstrap estimates are consistent with their counterpart parameters. Their study confirms the consistency; implying the essence of the portfolio analysis problem which was adequately captured by their proposed estimates. This greatly enhances the Markowitz mean-variance optimization model as being practically useful.

On the other hand Sharpe (1966) introduced the Sharpe ratio for the performance of mutual funds and portfolio selection. The Sharpe ratio is built on the Markowitz mean variance paradigm, which assumes that the mean and standard deviation of the distribution of one-period returns are sufficient statistics for evaluating the prospect of an investment portfolio (see e.g Sharpe, 1994). Since Sharpe introduced the ratio, most financial institutions have used it to evaluate the performance of mutual funds and select portfolio. Although, various measures have been proposed for evaluating portfolio performance (see e.g, Dowd, 2000, Campbell *et al.*, 2001), the Sharpe ratio is still a major index to measure the performance of mutual funds. Moreover, this ratio can be used to select the optimal portfolio on the efficient frontier generated by the Markowitz mean-variance model, because it considers both the mean and the standard deviation of the portfolio return. Specifically, fund managers can revise the objective function of the Markowitz mean variance model and then apply quadratic programming techniques to obtain the maximum Sharpe ratio portfolio.

The next is Tobin (1958) whose model was also based on the Markowitz's mean variance approach which led to the identification of a tangency portfolio, latter known as the market portfolio, along the efficient frontier (see e.g Fama and French, 2004, p.4). Tobin's model had a key assumption that cash was riskless asset (see Tobin, 1958, p.67). Hence when cash is added to the portfolio, the efficient frontier becomes a straight line. Assuming that investors are only concerned with the rate of return and the risk, an optimal portfolio would be somewhere along the straight line (see Campbell and Viceira, 2002, p.3).

The point at which the straight line touches the efficient frontier is known as the 'tangency portfolio', and it is the optimal mix of risky assets and riskless asset. Tobin's model is also referred to as the separation theorem, since the allocation of resources amongst risky asset is seen as a separate decision to the level of riskless asset with the portfolio.

However there have been serious criticisms of the Tobin's model, which are largely centered on the assumptions (see e.g Campbell and Viceira, 2002). It was observed that cash was not riskless in the long-run, because interest rate and inflation provide a return variance on cash. This variability implies risk as measured by the standard deviation. This would imply that in the long-run, the investor would select an optimal portfolio based on the mean-variance model precepts, which could have asset allocation significantly different from the short-run investors 'tangency portfolio'.

Farias *et al.*,(2006) investigated the comparison and performance of three portfolio selection models; Markowitz mean variance, Mean Absolute Deviation and Minimax models as applied to the Brazilian stock market (BOVESPA). For the purpose of the comparison, they used BOVESPA data from different 12 month time periods: 1999 to 2000, 2001 and 2002 to 2003. The first time period is typified by an up market, whereas the last two periods are dominated by down markets. Additionally, they evaluated the models' performance by using choice sets with different numbers of stocks available for investment. There are three choice sets: they are comprised of 20 stocks that the models can choose among them when making investments, another with 50 stocks, and the other with 100 stocks. This procedure is added to meet the diverse need of investors and may be a useful guide in their choice of portfolio selection models under different economic environment. Each model generated three different portfolios for each period, with performance determined by monthly returns over the period.

Although, the accumulated returns from the Minimax model were pretty much superior to the rest of the two, it was however observed that, the use of any of the three models was suitable during up markets.

Bower and Wentz (2005) also investigated the performance and the comparisons between the Markowitz mean variance model and Mean Absolute Deviation (MAD) model in portfolio optimization. As noted earlier, the computation of the Markowitz mean- variance approach calls for the use of covariance matrix, which becomes difficult to estimate for large portfolio. Konno and Yamazaki (1992) proposes alternative approach to the mean-variance model called the MAD model, which does not assume normality of the stock return as does the mean-variance of Markowitz. The MAD however minimizes a measure of risk as does the mean-variance, where the measure in this case is the Mean Absolute Deviation. MAD is easier to compute relative to Markowitz's mean-variance model because it eliminates the need for covariance matrix estimation. Bower and Wentz randomly selected 5 stocks and six-month bond from the S&P 500 for the study. Data covering six-month period were used for both models with a series of parametric and non-parametric test done on the data. They found that neither the mean-variance nor the mean absolute deviation model produced returns that are better than the other. They realized no statistically significant difference between the returns using both methods at the 5% level, but however observed some statistically significant difference at the 10% level. They concluded that with small portfolios, MV is the less complicated approach to use. However, since both returns using either method is not significantly different, they recommend in general that, it is acceptable to substitute MAD calculations for the MV method for small scale portfolios like 30 stocks. Meanwhile, they maintained that as the size of the portfolio increases, MAD model becomes increasing quicker to use. It is widely accepted that diversified portfolios results in best return while mitigating the risk level, both in the case of stocks and when stocks and bonds are combined (Markowitz, 2000). However, there has been little research into whether the same case applies for pure bond portfolios. Korn and Koziol (2006), Yawitz *et al*., (1976) indicate that diversification benefits exist in the case of pure bond portfolio. Ambrozaite and Sondergaard (2010) studied the Danish mortgage bond market to determine the highest possible return on bond investment for a unit of risk taken (i.e maximizing the Sharpe ratio). Data taken from the Danish bond market was analyzed with the Markowitz mean-variance approach. Sharpe ratios of individual bonds were compared to portfolios of various types of bond, including callable, non-callable and floating rate bonds. In addition the effect of short sales of bonds within the portfolio was assessed. They found that, combining the three types of bonds-callable, non-callable and the floating rate –in the portfolio yielded higher Sharpe ratios than portfolios consisting of only one or two distinct types of bond. They further concluded that investing in a portfolio of multiple bonds rather than individual bond dramatically reduces the risk (variance) while maintaining return. The diversification benefits were even more pronounced when short-selling of bonds was allowed in the portfolio.

Bonami and Lejeune (2009) studied the extension of the classical Markowitz mean variance portfolio optimization model. First, they considered that the expected asset returns are stochastic by introducing a probabilistic constraint, imposing that the expected return of the constructed portfolio exceed a prescribed return level with a high confidence level. They studied the deterministic equivalent of these models. In particular, they defined under which types of probability distributions the deterministic equivalents are second-order cone programs, and gave exact or approximate closed-form formulation. Secondly, they accounted for real-world trading constraints, such as the need to diversify the investments in a number of industrial sectors, the non-profitability of holding small positions, and the constraint of buying stocks by lots, modeled with integer variables. To solve the resulting problems, they proposed an *exact* solution approach in which the estimate of the expected return and the integer trading restrictions are *simultaneously* considered. The proposed algorithmic approach rests on a non-linear branch-and-bound algorithm which features two new branching rules. The first one is a static rule, called idiosyncratic risk branching, while the second dynamic, called portfolio risk branching. The study evaluated the efficacy of (4) four exact integer solution approaches on 36 problem instances containing up to 200 assets, and constructed using the stocks included in the S&P 500 index. They found that, any other computational study considering so many assets for a stochastic portfolio optimization model subject to integer constraints show that the solution approach using the portfolio risk branching rule is the most performing one, both in terms of speed and robustness.

Cesarone, Scozzari and Tardella (2009) also extended the original model of Markowitz by incorporating some real-world investment constraints into the model. Investment restrictions such as transaction cost, minimum lots sizes, complexity of management or policy of asset management companies, were termed as quality and cardinality constraints in the new model also known as the Limited Asset Markowitz (LAM) model which they proposed. The addition of these constraints results to a mixed integer quadratic programming problem, which is solve by reformulation of the model as a standard quadratic program. They tested their method with a 5 data set which include covariance matrices and expected return vectors of sizes ranging from 31 to 225 built from weekly price data covering a 5 year period for the Hang Seng, DAX, FTSE 100,S&P100, and Nikkei capital market indices. On these data sets, they were able to evaluate out-of-sample data, the performance of the portfolios obtained from the LAM model, and compared to the classical Markowitz MV portfolio selection, and the market index. Their comparison reveals that, solution obtained with the LAM was a better improvement to the Markowitz model when some real-world investment constraints were introduced.

Levy and Ritou (2001) also investigated the properties of mean-variance efficient portfolios when the number of assets is large. They analytically and empirically demonstrated that the proportion of assets held short converges to50% as the number of assets grows, and the investment proportions are extreme, with several assets held in large positions, the cost of the no-short selling constraint increase dramatically with the number of asset. They also found that, for 100 assets, the Sharpe ratio can be more than doubled with the removal of this constraint. These results seem to be fundamental properties of mean-variance efficient portfolios in large market.

In a comparative study of the Markowitz model and the Sharpe's model, Affleck-Graves and Money (1976) noted interesting link between the two models. Their study used the expected index portfolio return and standard deviations, and observed that the result obtained with the Sharpe's model became progressively better with every index that was added. It further noted that if more portfolios are added to the point that each share was its own portfolios, the model simulates the Markowitz model. Again, it was found that if very low upper boundaries (in terms of percentage holding of any one share) were enforced on Markowitz model, the single-index model was a close approximation of the optimal portfolio. The study also found that

Markowitz model naturally limits the maximum weight invested in any one share to about 40 percent (if no upper boundaries were enforced) and has in the region of six shares in the efficient portfolio which they felt gave it a natural diversification. In its simplest form the Markowitz model states that a portfolio that will give a minimum variance for a target expected return can be unambiguously selected from the collection of assets. In other words, for every possible target portfolio return, there is a unique portfolio of assets that will give the required return at a minimum variance.

In conclusion, mean-variance optimization has under the banner of modern portfolio theory (see for example, Rudd and Clasing 1982), gained widespread acceptance as a practical tool for portfolio construction. This has occurred over the last decade primarily as a result of the technological advances made in estimating covariance of portfolio return. Many investment advisory firms and pension plan sponsors (and their consultants) today routinely compute mean variance efficient portfolios as part of the portfolio allocation process. Specific applications include asset allocation (allocation across the broad asset classes such as stock and bonds), multiple money managers decisions (allocation across money manager with different strategies and objectives), index matching (finding a portfolio whose returns will closely track those of a predetermined index such as the S&P 500), and active portfolio management (optimizing risk-return tradeoff assuming superior judgment).

CHAPTER 3

3.0 METHODOLOGY

3.1 Mathematics of the Markowitz Model.

The Markowitz model involves some mathematics, which make it possible to construct stock portfolios with different combinations where short sale and lending or borrowing might be allowed or not. The Markowitz model is all about maximizing return, and minimizing risk, but simultaneously. We should be able to reach a single portfolio of risky assets with the least possible risk that is preferred to all other portfolio with the same level of return. Our optimal portfolio will be somewhere on the ray connecting risk free investment to our risky portfolio and where the ray becomes tangent to our set of risky portfolios. This point has the highest possible slope. Markowitz uses the arithmetic Mean , the variance and the covariance parameters for return and risk estimations. The model is a form of quadratic programming problem.

3.1.1 Mean calculation

The mean is a measure of an average return of a portfolio. The mean of a portfolio can be calculated with several methods, but mainly arithmetic and geometric. In this work, we have chosen arithmetic for our analysis. Let us look at them briefly.

Definition 3.1.2 Arithmetic Mean.

The arithmetic mean of a list of numbers (observations) is the sum of all the members of the list divided by the number of items in the list.

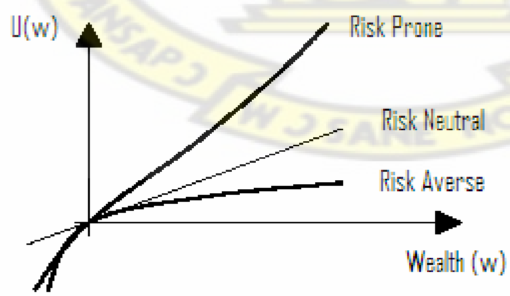
Where

Then

Example 1 ***Finding optimal level of portfolio returns***

Suppose that an institution has a portfolio of three securities

a)



3.2 Formulation of the Markowitz Model.

Let us revisit the assumptions of the model; which are:

- Investors seek to maximize the expected return of a total wealth.
- All investors have the same expected single period investment horizon.

- All investors are risk averse, ie they will only accept a higher risk if they are compensated with a higher expected return.
- Investors base their investment decision on the expected return and risk.
- All markets are perfectly efficient; ie all available information on the market reflect on the security prices.
- Short-selling strategy is not permitted.

3.2.1 Return

Let

Where

Therefore, the return on a market index, defined in theory as the weighted mean of all the securities that make up the index with the weightings being obtained from the market capitalization of each security, which is calculated in practice, by using the value of the indices quoted on the markets directly.

Thus,

Intuitively, asset risk is characterized by the dispersion of the asset return around their average value. The statistical measurements are therefore the variance

•

Max

3.2.3 Formulation Equivalence

Equation (4.0) maximizes a (concave) linear function subject to quadratic and linear constraints; while equations (4.1) and (4.2) minimize convex quadratic function subject to linear constraints. When problem (4.2) gives us the minimum-variance portfolio without considering the expected return.

3.3 Diversification.

The variance expression in equation (4.0) reveals the usefulness of diversification in risk reduction, which could be derived as follows;

Suppose the assets are all independent, in particular, they are uncorrelated, so

The limit is the average covariance which is the measure of the non-diversifiable market risk

The figure 1.2 below depicts how total risk diminishes as more randomly selected common stocks are added to the portfolio. But when more than about three dozen random stock are added, it is impossible to reduce a randomly selected portfolio risk below the level of non-

diversifiable risk that exist in the market. The straight line separates the systematic risk from the unsystematic one. The non-diversifiable risk lies below the straight line.

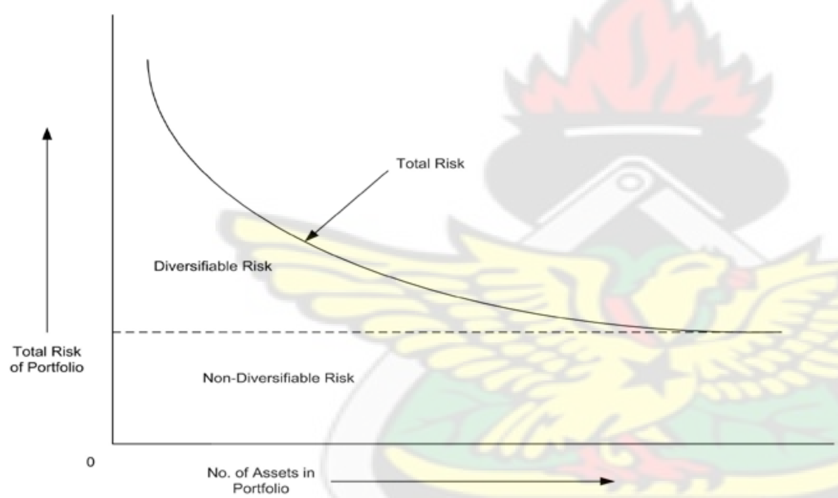


Figure 1.2- the effect of number of securities on risk of the portfolio.

Source: Adapted from Marx, Mpofu and Van de Venter (2003, p. 38).

Definition 3.4.1 Efficient portfolio.

Let

3.4 The Markowitz efficient frontier

Every possible asset combination can be plotted in a risk-return space, and the collection of all such possible portfolios defines a region in this space. The line along the upper edge of this region is known as the efficient frontier. Combinations along this line represent portfolio (explicitly excluding the risk- free alternative) for which there is a lowest risk for a given level of return, or conversely, for a given amount of risk, the portfolio lying on the efficient frontier represents the combination offering the best possible return. Mathematically, the efficient frontier is the intersection of the set of portfolios with minimum variance and the set of portfolios with maximum return.

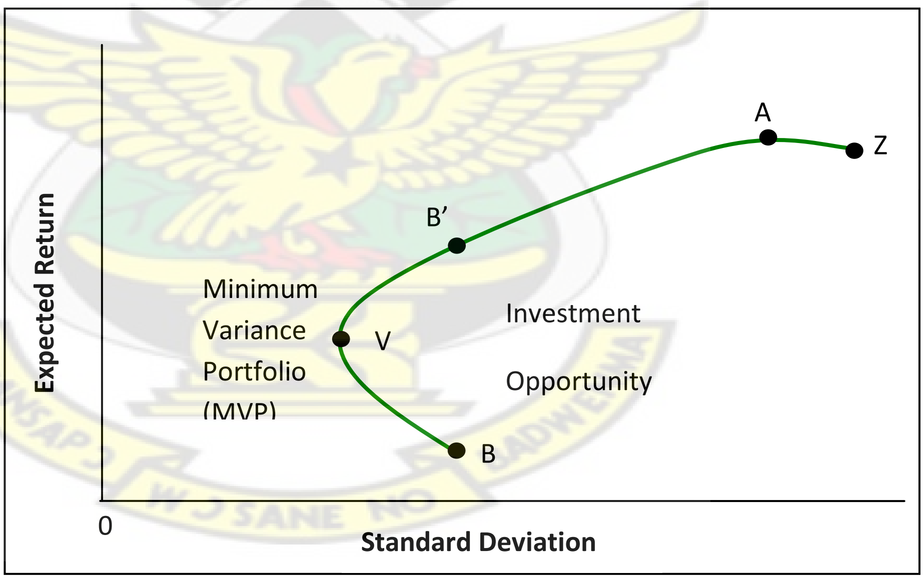


Figure1.3a Investment opportunity set for asset A and asset B

Source: Chung, et al(2009,p.9)

Figure 1.3a above shows the entire investment opportunity set, which is the set of all attainable combinations of risk and return offered by asset A and B in different proportions. Investors desire portfolios that lie to the northwest in fig 1.3a. These are portfolios with high return and low volatility.

The area within the curve BVAZ is the feasible opportunity set representing all possible portfolio combinations. Portfolios that lie below the minimum-variance portfolio (point V) on the curve can therefore be rejected as being inefficient. The portfolios that lie on the frontier VA would not be likely candidates for investor to hold, since the portfolios do not meet the criteria of maximizing expected return for a given level of risk, or minimizing risk for a given level of return. This is easily seen by comparing the portfolio represented by points B and o

3.4.2 No- Correlation between assets

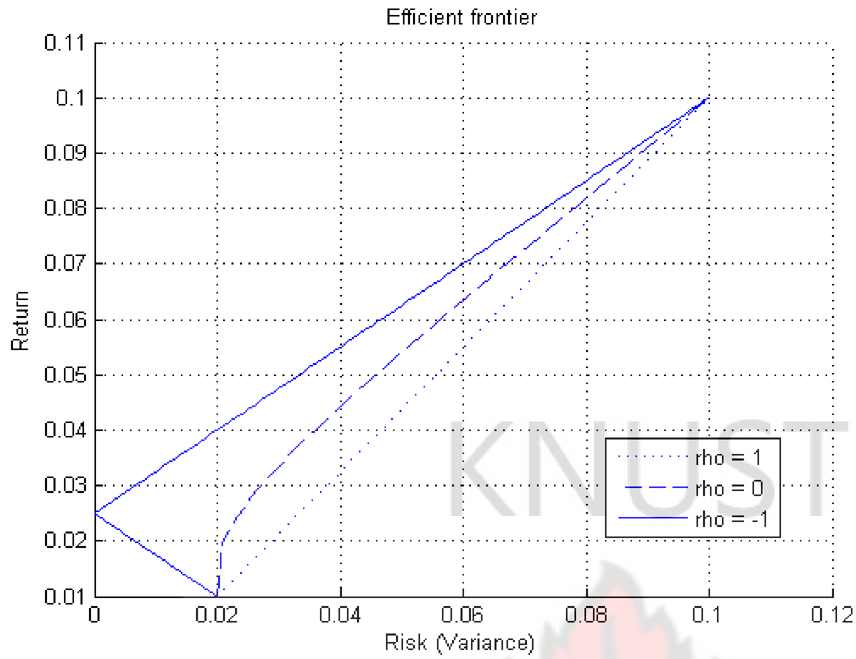


Figure 1.3b, the graph of positive, negative and non-correlation of two assets.

Source: Roudier, F(2007,p.12)

Because of correlations between assets, it is possible to build a portfolio less risky than all individual assets, but with a higher expected return than the lowest expected return among these assets. For a given level of expected return

3.5 Minimum variance portfolio

Suppose an investor desires to invest in a portfolio with the least amount of risk. He does not care about his expected return; he only wants to invest all his money in a portfolio with the lowest possible amount of risk. Because he will always invest in an efficient portfolio, he will choose a portfolio on the efficient frontier with minimum standard deviation. At this point, the variance is also minimal, that is why this portfolio is called minimum variance portfolio. This minimum variance portfolio can be calculated by minimizing the variance subject to the constraint that the investor can only invest the amount of capital he has (i.e the budget constraint).

Mathematically, the optimization problem is thus,

Min

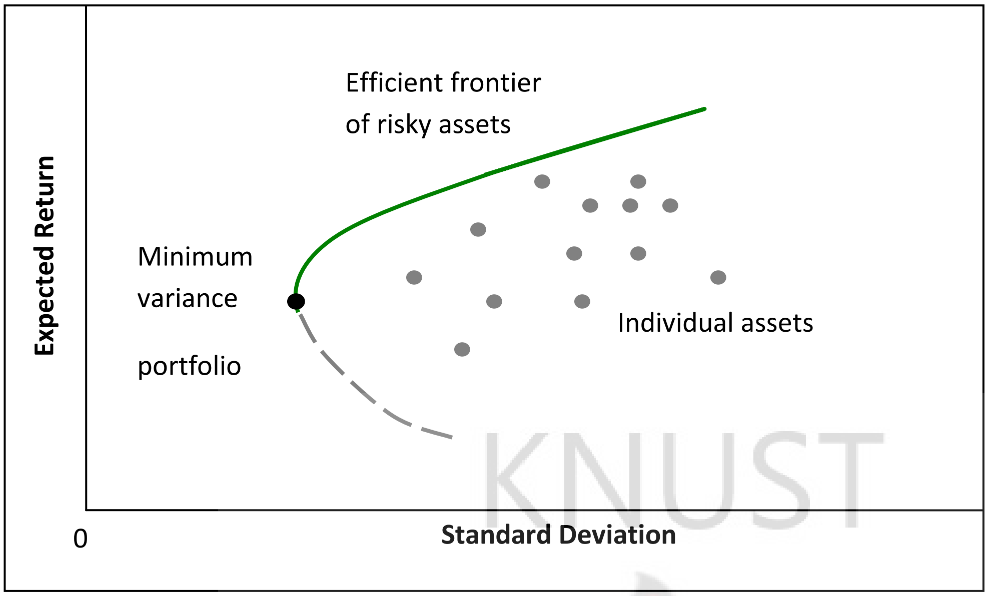
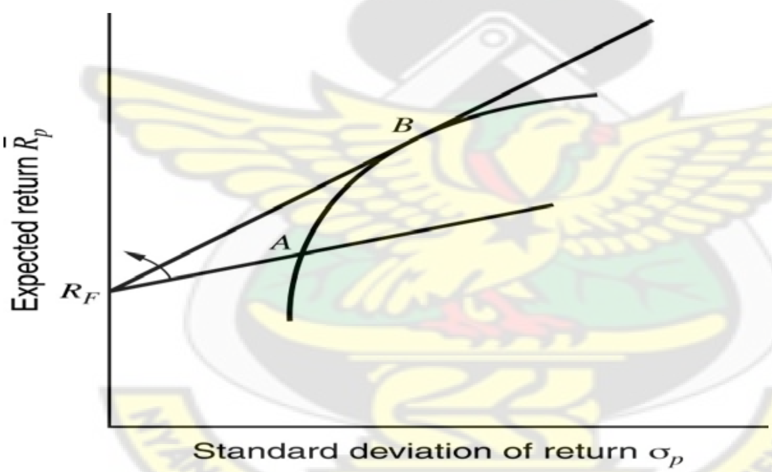


Fig.1.4 The minimum variance portfolio

Source: Chung, et al(2009, p.10)

3.6 The Tangency portfolio

Suppose an investor has other preferences than taking the least possible amount of risk (thus investing in minimum variance portfolio), but rather investing in the portfolio with maximum Sharpe ratio. The Sharpe ratio is defined as the return-risk ratio, thus:


Expected return $\bar{R}_p$
R_F
A
B
Standard deviation of return σ_p

Mathematically, the optimization problem is given by:

Max

Since equation (4.6a) is a product of two functions, we need to use both the product rule and the chain rule.

The solution of equation (4.8) involves solving the following system of simultaneous equations.

Table 1.1

	Rate of return 1	Rate of return 2	Rate of return 3
Stock A	5%	10%	15%
Stock B	2%	20%	22%
Stock C	0%	20%	30%
Probability	0.3	0.6	0.1

Solution

The expected rate of return is given by:

CHAPTER 4

DATA AND ANALYSIS

Data used in this study were obtained from the Ghana Stock Exchange (GSE). The four year period data spanning from 2007 to 2010 comprise of market capitalization, stock prices and divided per share, out of which the Financial, Non-Financial, and the All-Stock Indices were computed. To do this we first find the return on the stocks or the security

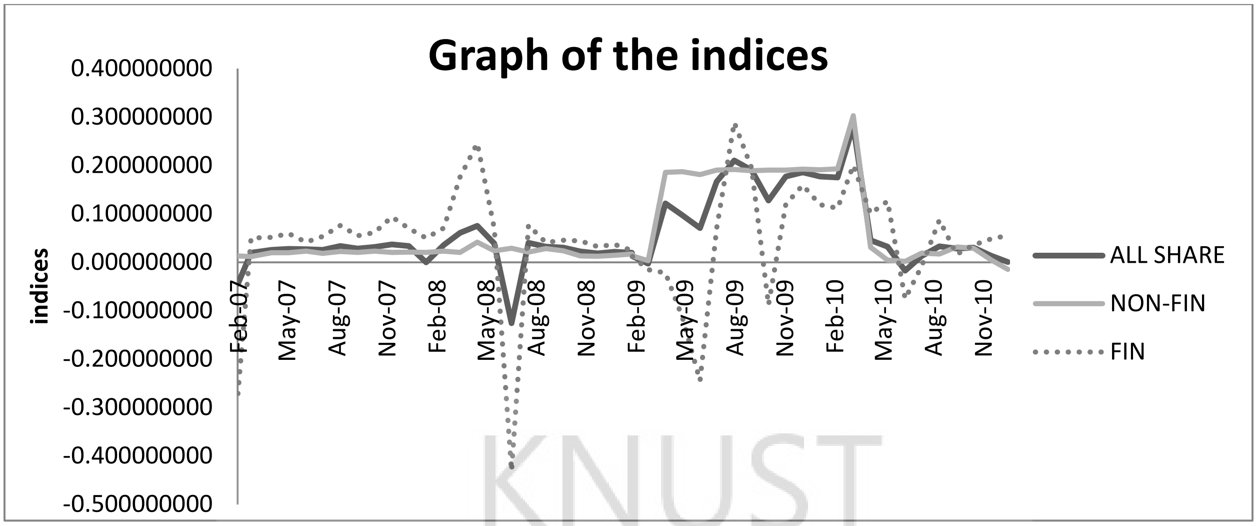


Figure 1.6 monthly indices

We can project the evolution of returns on the index if an investor put Ghc100.00 in the three portfolios for the next 48 months starting from January 2007 to December 2010.

The time series is given by:

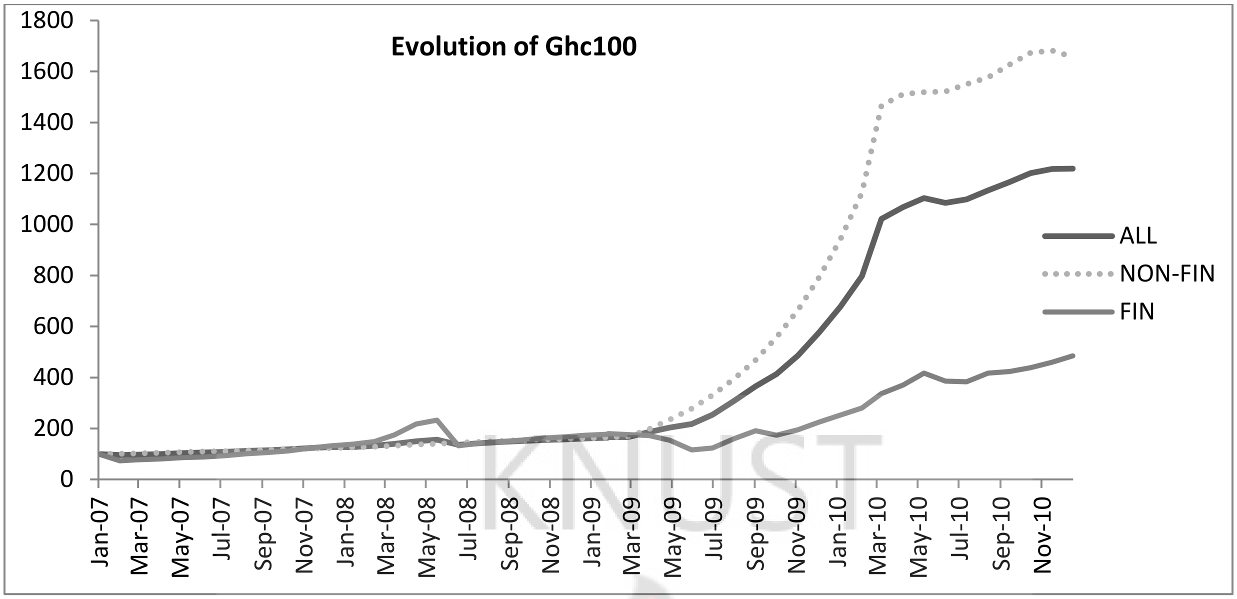


Figure 1.7 return evolution

From figure 1.7 we observe that the investor earns more return in the non-financial index at the end of the period; and hence he should allocate more of his capital in the non-financial index.

The following table 1.3 is the mean return data for the All-share, Non-financial and the Financial Index with the variances.

Table 1.3: mean return and variance of All-Share, Financial and Non-financial index

Statistics	ALL-SHARE INDEX	NON-FINANCIAL INDEX	FINANCIAL INDEX
Mean	0.057142419 (5.71%)	0.064438907 (6.44%)	0.042590073 (4.26%)
t-stat	5.230348	5.411107	2.378557
Variance	0.005609884 (0.56%)	0.006665336 (0.67%)	0.015069108 (1.51%)

We perform t-test to see whether the mean return is significantly different from 0. From the test, we saw that the mean return was statistically significant from 0.

From table 1.3 above, we can clearly conclude that, it is advisable for an investor to invest in the non-financial index since it has the highest return of 6.44% with relatively lesser risk of 0.67% compared to the Financial and the all-share index.

The variance/covariance matrix

	ALL-SHR	NON-FIN	FIN'CIAL
ALL-SHR	0.005491	0.005343	0.005187
NON-FIN	0.005343	0.006524	0.001594
FIN'CIAL	0.005187	0.001594	0.014748

4.1 Diversification

The correlation matrix is also given by:

	ALL-SHR	NON-FIN	FIN'CIAL
ALL-SHR	1.00	0.892763	0.576365
NON-FIN	0.892763	1.00	0.162529
FIN'CIAL	0.576365	0.162529	1.00

From the correlation matrix above, it is clear that, there exist a positive correlation between the portfolios, which makes diversification relatively difficult to reduce portfolio risk. Even though the correlation between the three portfolios is positive, we see that the correlation between the Non-financial and the Financial indices is lesser, 16.25%, relative to the correlations that exist between the All-stock and the Non-Financial, the All-stock and the Financial. This allows some degree of diversification to be undertaken by investors.

Nonetheless, the total risk of the portfolio is largely due to the market risk or the non-diversifiable risk, since there is no much unique risk to be diversified away due to the presence of positive correlation among assets.

4.2 Solving the Optimization Problem

Our optimization problem is a case of three assets scenario, where All-stock= X_1, Non-Financial stock=X_2, and Financial stocks=X_3, which represent the weight on each asset, and by the Markowitz assumption, $X_1+X_2+X_3=1$.

Again, the non-short- selling constraint requires that X_1, X_2, and X3 > 0.

Now, we look for a minimum variance portfolio which is given by:

Min

The optimization problem in excel solver is given as follows;

	MEAN	VARIANCE
ALL	0.0571424	0.0056099
NON-FIN	0.0644389	0.0066653
FIN	0.0425901	0.0150691

covariance matrix

		0.3333333	0.3333333	0.333333
		ALL-SHR	*NON-FIN*	*FIN'CIAL*
0.33333	ALL-SHR	0.005491	0.005343	0.005187
0.33333	NON-FIN	0.005343	0.006524	0.001594
0.33333	FIN'CIAL	0.005187	0.001594	0.014748
1.00000		0.00178	0.001496	0.002392
0.054724	Mean			
0.075285	SD			
0.726892	Slope			

cell to store constraint on risk premium 0.08000

	EQ	min var					**Optimum**	
Mean	0.054724	0.058483	0.059	0.06	0.0603	0.0606	**0.060821**	0.064439
SD	0.075285	0.071971	0.072042	0.072574	0.072835	0.073141	**0.073396**	0.080768
Slope	0.726892	0.812591	0.818972	0.82674	0.827902	0.828537	**0.828675**	0.797824
All	0.333333	0	0.00000	0	0	0	**0.00000**	0
Non-Fin	0.333333	0.727415	0.75107	0.796836	0.810566	0.824297	**0.83443**	1
Fin	0.333333	0.272585	0.24893	0.203164	0.189434	0.175703	**0.16557**	0
CAL*	0.062387	0.059641	0.059699	0.06014	0.060356	0.06061	0.060821	0.066931

risk premium on CAL=sd*slope of optimal risky portfolio

We generate points on the efficient frontier as shown above. At the minimum variance portfolio (min var column), we obtain a mean returns of 5.84%, minimum standard deviation or risk of 7.19% with the Sharpe ratio of 81.26%. Under this minimum variance portfolio selection, an investor is supposed to invest 72.74% of his total budget into the Non-financial stocks, 27.25% in the financial stock. Note take at this point nothing shall be invested into the all-share, since the all-share index is the summation of the financial and the non-financial indices. Recall also that the Sharpe ratio measures the reward to risk variability; therefore, the higher the Sharpe ratio, the better the portfolio selection for the investor. In our above optimization problem, the risk-free rate is assumed to be zero.

However, if the investor chooses to invest in an optimum portfolio, thus by maximizing the Sharpe ratio, he obtains mean returns of 6.08%, and risk of 7.34%, with a maximum Sharpe ratio of 82.87%. More so, he needs to invest 83.44% in non-financials and 16.56% in the financials. If the investor wishes to further increase his expected return above the optimum portfolio level as indicated above, or even prefers a corner solution by investing all his budget or allocate 100% of his budget in the non-financial index, his reward to risk volatility reduces to 79.78%, though feasible, but non-optimal.

We also observe the inverse relationship between the weights invested in the non-financial and the financial index. As the investor seeks for an optimum portfolio, he invests more in the non-financials but less in the financial index. This is better explained with the following graph.

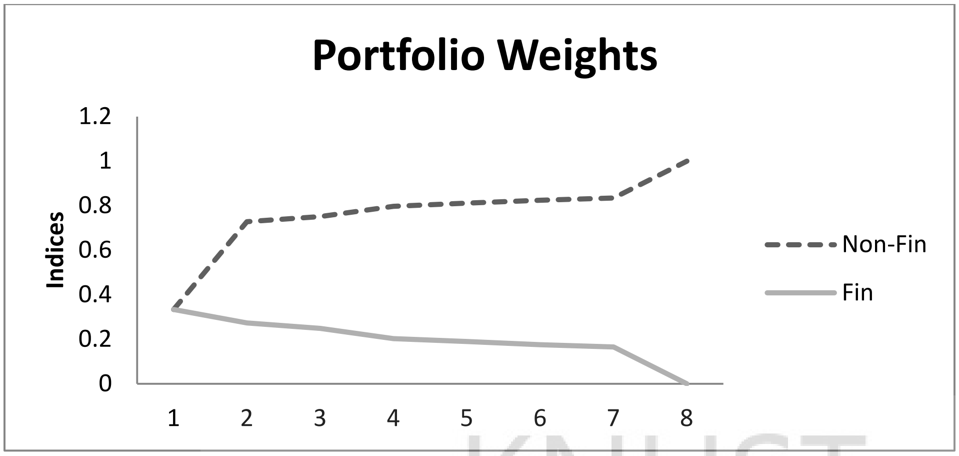


Figure 1.8 inverse relations between financial and non-financial weights.

Figure 1.8 shows that, as the investor choose to invest more in the non-financial index, the amount of capital invested in the financial index approaches zero.

4.3 The Efficient Frontier.

The combination of risk-return possibilities can be plotted in a risk-return space. The line joining these points is called the efficient frontier. Any point on this curve represents portfolio (explicitly excluding the risk-free rate) offering the best possible return, for a given level of risk.

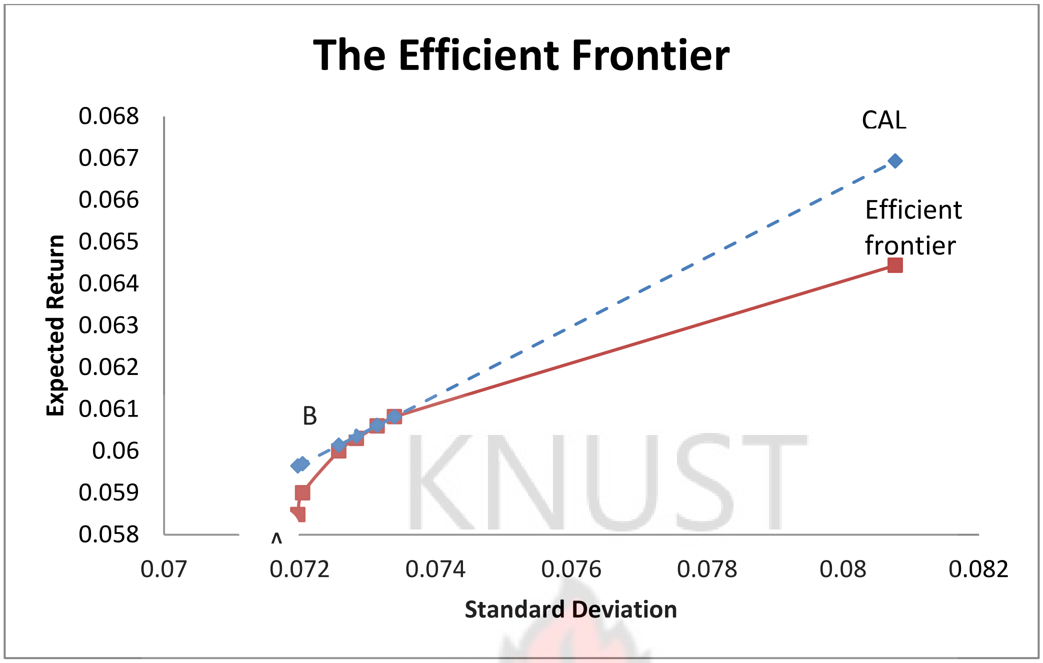


Figure 1.9 the investment possibility frontier.

A risk averse investor would select a portfolio at point **A** (0.072, 0.058) which represent a minimum variance portfolio, while a risk lover would select a portfolio at point **B** (0.072, 0.061) which represent the optimum portfolio from figure 1.9 above. The optimum portfolio is also known as the tangency portfolio, because at this point the capital allocation line (CAL) is tangent to the efficient frontier. The CAL has a slope equal to the Sharpe ratio of the optimal risky portfolio. The CAL values are obtained by multiplying the standard deviation of each portfolio by the Sharpe ratio of the optimal risky portfolio.

CHAPTER 5

CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

We have so far examined portfolio selection under the Markowitz mean variance approach where short-selling is not allowed. We applied the model on the stock indices of the Ghana Stock Exchange; namely, financial index, non-financial index and the all-share index. We observed that the best investment opportunities that yield higher return of 6.08% for a given level of risk, is to invest 83.44% of his budget into the non-financial index and 16.56% in the Financial sector during the period under review. We must however sound a caveat that, the Non-financial index has been falling steadily, while the financial index maintains a rather steady and surprising growth. We further observed that diversification reduces portfolio risk, especially the unique or unsystematic risk of the asset. Diversification however becomes relatively difficult task when the correlation between assets or stocks approaches to unity or perfect positive correlation.

These results and observations contribute significantly to the existing knowledge on the Ghanaian stock market since an average investor can now select his portfolio largely on the non-financial index. We conclude that the Ghana Stock Exchange obeys the tenets of the classical Markowitz mean variance model.

5.2 Recommendations

We have seen from the previous section that, the most performing index during the year under review is the Non-financial, hence the recommendation is that current investors and prospective ones on the Ghana stock market should allocate more of their resources in the Non-financial index.

Recommendations for areas of further research include the allowance of trading restrictions such as the minimum amount of capital to invest in an asset, the requirement to buy assets in large lots, transaction cost and permission of short-selling in the original Markowitz model. Also the Markowitz mean variance approach should be studied in the light of bond and mutual fund portfolio selection on the Ghanaian capital market.

REFERENCES

Afflec-Grave,J.F and Money, A.H (1976) A Comparison of Two Portfolio Selection Models. The Investment Analyst Journal.7 (4):35-40.

Ambrozaite, R. and Sondergaard, L. (2010) . Danish Mortgage bond portfolio optimization using the mean-variance approach. Master's thesis, Copenhagen Business School.

Amenc, N and Veronique, L (2003), Portfolio Theory and Performance Analysis,2nd ed.

Bai,.Z. Liu, H. and Wong, W.K (2007), 'Making Markowitz's Portfolio Optimization Theory Practicably Useful'

Bower, B and Wentz, P (2005)Portfolio Optimization Mean-Absolute Deviation vs. Markowitz.

Campbell, J.Y and Luis M.VB (2004) "Long-Horizon Mean-Variance Analysis: A user Guide", Manuscript, Havard University, Cambridge, MA.

Cesarone,.F., Scozzari,.A. and Tardella, F (2009), Algorithms for constrained portfolio optimization. Master's thesis, Universita di Roma "La sapienza"

Chin, W. Yang, Ken Hung(2010); A generalized Markowitz portfolio selection model with Higher moments. Master's thesis.

Donkor, M (2011), Optimal Portfolio Using Markowitz. Master's thesis, Kwame Nkrumah University of Science and Technology, Ghana.

Ehrgott, M., Klamroth, K. and Schwehm, C. (2004) An MCDM Approach to portfolio optimization, European Journal of operational Research, 155,752-70.

Elliot, R.J and Kopp, P.E(1999), Mathematics of financial markets, Springer New York.

Elton, E.J., and Gruber, M.J.,(1997). Journal of Banking and Finance 21(1997):1743-1759.

Elton, E.J and Gruber, M.J (1987) ‘Modern Portfolio Theory and Investment Analysis, John Wiley and Sons,3rd edition, New York.

Engels, M (2004), Master’s Thesis, Portfolio Optimization beyond Markowitz, University of Leiden, The Netherlands.

Evans,J.L and Archer, S.H (1968); ‘Diversification and the reduction of dispersion: An empirical analysis’ Journal of finance 761-767.

Grootveld,.H and Hallerbach, W(1999). Variance versus Downside risk: Is there really much difference? European journal of operational research,114,304-9.

Huang,C.F and Litzenberger, R.H(1988). Foundations for financial Economics, North-Holland, New York.

Jorion, P.(2003). Portfolio optimization with tracking error constraints; Financial analyst Journal 59,70-82

Kheirollah, A and Bjarnbo, O (2007) A Quantitative Risk Optimization of Markowitz Model: An Empirical Investigation on Swedish Large Cap List.

Kroll,. Y and Markowitz, H.M (1984). Mean –variance versus direct utility maximization, Journal of finance ,39-47-61.

Konno,.H. and Yamazaki,.H.(1991). Mean Absolute deviation portfolio optimization model and its application to Tokyo Stock Exchange, Management Science,37(1991)519-531

Li, K (2008), Continuous-Time Mean-Variance Portfolio Selection. Master's thesis, University of Oxford, UK.

Maharakkhaka, B (2011) The Performance of mean variance portfolio selection and its opportunity cost: The case of Thai Securities.

Markowitz, H. (1952). "Portfolio Selection" Journal of Finance, Vol.7 no.1 March: 77-91

Markowitz, H. (1959). "Portfolio Selection": Efficient Diversification of Investment.

Merton, R. C (1969). "Life time Portfolio Selection Under Uncertainty: The Continuous-Time Case". Review of Economics and Statistics, Vol.51 no.3 (August): 247-257.

Mwambi.S. and Mwamba.M (2010). An alternative to portfolio selection problem beyond Markowitz's Log optimal growth portfolio.Master's thesis,University of Johannesburg. South Africa.

Nagurney, A (2009) 'Portfolio Optimization. Master's thesis, Havard University Graduate school of Design.

Plesis A.J, and M. Ward(2009). Applying the Markowitz portfolio selection model as a passive investment strategy on the Johannesburg Stock Exchange. Master's thesis, University of Johannesburge, South Africa.

Sharpe, W.F (1994). The Sharpe ratio, Journal of portfolio management, 21-49-58.

Statman, M (2001), "How many stocks make a diversified portfolio?" Journal of Finance and Quantitative Analysis. Manuscript, Harvard University, Cambridge, MA.

Ulucan, A.(2007). An analysis of mean variance portfolio selection with varying holding periods, Applied Economics,39,1399-407.

West, G., (2006), An introduction to Modern Portfolio Theory: Markowitz, CAPM, APT and Black-Literman.

2011 International Conference on Economic and Financial Research IPEDR vol.4 (2011), LACSIT press, Singapore. Quantitative Finance and Risk Management.