MICROFINANCE LOAN PORTFOLIO MANAGEMENT

(A CASE STUDY OF WESTERN MICROFINANCE LTD BASED IN TAKORADI)

BY

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DECLARATION

I Frederick Kirk Duku hereby declare that except for reference to other people's work, which have duly been cited. This submission is my own work towards the Master of Science degree and

that, it contains no material previously published by another person nor presented elsewhere.

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DEDICATION

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ABSTRACT

The numbers of Microfinance Institutions (MFIs) in the country have increased in recent years, however, they tend to go down or even collapse when the recovery rate on loan is low or the number of defaults goes up in economic down turns.

This phenomena has led to most Microfinance Institutions adopting cutting edge technology to

Improve the quality of their Loan structure. The decline of relevant portfolio planning Models especially in Ghana is attributed mainly to the evolving dynamics of the Ghanaian

Microfinance industry where the regulatory controls have changed with a high frequency.

The purpose of this Study is to propos a linear model and solved using the revised simplex method which will maximize Western Microfinance Limited profit on loans and allocate their funds for loan disbursement leading to financial sustainability.

The results from the model showed that Western Microfinance Limited would make a profit

Of GH¢46930.00 if they are to stick to the model. From the study, it was realized that the scientific method used to develop the propose model can have an increase in profit margin if put into practice.

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CHAPTER 1

INTRODUCTION

1.1 BACKGROUND OF THE STUDY

Microfinance is the provision of financial services to low-income clients or solidarity lending groups including consumers and the self-employed, who traditionally lack access to banking and related services.

More broadly, it is a movement whose object is "a world in which as many poor and near-poor households as possible have permanent access to an appropriate range of high quality financial services, including not just credit but also savings, insurance, and fund transfers"(Robert et al. ,2004). Those who promote microfinance generally believe that such access will help poor people out of poverty.

Microfinance is a broad category of services, which includes microcredit. Microcredit is provision of credit services to poor clients. Although microcredit is one of the aspects of microfinance, conflation of the two terms is epidemic in public discourse. Critics often attack microcredit while referring to it indiscriminately as either 'microcredit' or 'microfinance'. Due to the broad range of microfinance services, it is difficult to assess impact, and very few studies have tried to assess its full impact (Feigenberg et al. ,2011).

Microfinance and Microcredit

In the literature, the terms microcredit and microfinance are often used interchangeably, but it is important to highlight the difference between them because both terms are often confused. Sinha (1998,p.2) states "microcredit refers to small loans, whereas microfinance is appropriate where NGOs and MFIs supplement the loans with other financial services(savings, insurance , etc)". Therefore microcredit is a component of microfinance in that it involves providing credit to the poor, but microfinance also involves additional non-credit financial services such as savings insurance, pensions and payment services (Okiocredit, 2005)

The History of Microfinance

Microcredit and microfinance are relatively new terms in the field of development, first coming to prominence in the 1970s, according to Robinson(2001)and Otero(1999). Prior to then, from the 1950s through to the 1970s, the provision of financial services by donors or governments was mainly in the form of subsidized rural credit programmes. These often resulted in high loan defaults, high loses and an inability to reach poor rural households (Robinson, 2001).

Robinson states that the 1980s represented a turning point in the history of microfinance in that MFIs such as Grameen Bank and BRI began to show that they could provide small loans and savings services profitably on a large scale. They received no continuing subsidies, were commercially funded and fully sustainable, and could attain wide outreach to clients (Robinson, 2001). It was also at this time that the term "microcredit" came to prominence in development (MIX, 2005). The difference between microcredit and the subsidized rural credit programmes of the 1950s was that microcredit insisted on repayment, on charging interests rates that covered the cost of credit delivery and by focusing on clients who were dependent on the informal sector for credit (ibid). It was now clear for the first time that microcredit could provide large-scale-outreach profitably.

The 1990s "saw accelerated growth in the number of microfinance institutions created and an increased emphasis on reaching scale" (Robinson, 2001, p.54). Dichter (1999, p 12) refers to the 1990s as "the microfinance decade". Microfinance had now turned into an industry according to Robinson (2001).

Along with the growth in microfinance institutions, attention change from just the provision of credit to the poor (microcredit), to the provision of other financial services such as savings and pensions (microfinance) when it became clear that the poor had a demand for these other services (MIX, 2005).

The importance of microfinance in the development was reinforced with the launch of the Microfinance Summit in aims to reach 175 million of the world's poorest families, especially the women of those families, with credit for the self-employed and other financial and business services, by the end of 2015 (Microcredit Summit, 2005). More recently, the UN, as previously stated, declared 2005 as the international Year of Microcredit.

Challenges

Traditionally, banks have not provided financial services, such as loans, to clients with little or no cash income. Banks incur substantial costs to manage a client account, regardless of how small the sums of money involved. For example, although the total gross revenue from delivering one hundred loans worth $1,000 each will not differ greatly from the revenue that results from delivering one loan of $100,000, it takes nearly a hundred times as much work and cost to manage a hundred loans as it does to manage one. The fixed cost of processing loans of any size is considerable as assessment of potential borrowers, their repayment prospects and security; administration of outstanding loans, collecting from delinquent borrowers, etc., has to be done in all cases. There is a break-even point in providing loans or deposits below which banks lose money on each transaction they make. Poor people usually fall below that breakeven point. A similar equation resists efforts to deliver other financial services to poor people.

In addition, most poor people have few assets that can be secured by a bank as collateral. As documented extensively by Hernando de Soto and others, even if they happen to own land in the developing world, they may not have effective title to it.This means that the bank will have little recourse against defaulting borrowers.

Seen from a broader perspective, the development of a healthy national financial system has long been viewed as a catalyst for the broader goal of national economic development (see for example Alexander Gerschenkron, Paul Rosenstein-Rodan, Joseph Schumpeter, Anne Krueger). However, the efforts of national planners and experts to develop financial services for most people have often failed in developing countries, for reasons summarized well by Adams, Graham & Von Pischke in their classic analysis 'Undermining Rural Development with Cheap Credit(Adams et al.1984) .

Because of these difficulties, when poor people borrow they often rely on relatives or a local moneylender, whose interest rates can be very high. An analysis of 28 studies of informal moneylending rates in 14 countries in Asia, Latin America and Africa concluded that 76% of moneylender rates exceed 10% per month, including 22% that exceeded 100% per month. Moneylenders usually charge higher rates to poorer borrowers than to less poor ones (Marguerite Robinson, 2001). While moneylenders are often demonized and accused of usury, their services are convenient and fast, and they can be very flexible when borrowers run into problems. Hopes of quickly putting them out of business have proven unrealistic, even in places where microfinance institutions are active.

Over the past centuries practical visionaries, from the Franciscan monks who founded the community-oriented pawnshops of the 15th century, to the founders of the European credit union movement in the 19th century (such as Friedrich Wilhelm Raiffeisen) and the founders of the microcredit movement in the 1970s (such as Muhammad Yunus) have tested practices and built institutions designed to bring the kinds of opportunities and risk-management tools that financial services can provide to the doorsteps of poor people (Helms and Bright,2006). While the success of the Grameen Bank (which now serves over 7 million poor Bangladeshi women) has inspired the world, it has proved difficult to replicate this success. In nations with lower population densities, meeting the operating costs of a retail branch by serving nearby customers has proven considerably more challenging. Hans Dieter Seibel, board member of the European

Microfinance Platform, is in favour of the group model. This particular model (used by many Microfinance institutions) makes financial sense, he says, because it reduces transaction costs. Microfinance programmes also need to be based on local funds.

Although much progress has been made, the problem has not been solved yet, and the overwhelming majority of people who earn less than $1 a day, especially in the rural areas, continue to have no practical access to formal sector finance. Microfinance has been growing rapidly with $25 billion currently at work in microfinance loans (Deutsche Bank,2007). It is estimated that the industry needs $250 billion to get capital to all the poor people who need it (Deutsche Bank,2007).

The industry has been growing rapidly, and concerns have arisen that the rate of capital flowing into microfinance is a potential risk unless managed well (www.citigroup.com/citi/microfinance/data/initiatives.pdf).

As seen in the State of Andhra Pradesh (India), these systems can easily fail.Some reasons being lack of use by potential customers, over-indebtedness, poor operating procedures, neglect of duties and inadequate regulations (www.inwent.org/ez/articles/184683/index.en.shtml).

Boundaries and principles

Poor people borrow from informal moneylenders and save with informal collectors. They receive loans and grants from charities. They buy insurance from state-owned companies. They receive funds transfers through formal or informal remittance networks. It is not easy to distinguish microfinance from similar activities. It could be claimed that a government that orders state banks to open deposit accounts for poor consumers, or a moneylender that engages in usury, or a charity that runs a heifer pool are engaged in microfinance. Ensuring financial services to poor people is best done by expanding the number of financial institutions available to them, as well as by strengthening the capacity of those institutions. In recent years there has also been increasing emphasis on expanding the diversity of institutions, since different institutions serve different needs.

Some principles that summarize a century and a half of development practice were encapsulated in 2004 by Consultative Group to Assist the Poor (CGAP) and endorsed by the Group of Eight leaders at the G8 Summit on June 10, 2004 (Helms, 2006):

1. Poor people need not just loans but also savings, insurance and money transfer services.

2. Microfinance must be useful to poor households: helping them raise income, build up assets and/or cushion themselves against external shocks.
3. "Microfinance can pay for itself"(Helms, 2006). Subsidies from donors and government are scarce and uncertain, and so to reach large numbers of poor people, microfinance must pay for itself.
4. Microfinance means building permanent local institutions.
5. Microfinance also means integrating the financial needs of poor people into a country's mainstream financial system.
6. "The job of government is to enable financial services, not to provide them"(Helms, 2006).
7. "Donor funds should complement private capital, not compete with it" (Helms, 2006).
8. The key bottleneck is the shortage of strong institutions and managers" (Helms, 2006). Donors should focus on capacity building.
9. Interest rate ceilings hurt poor people by preventing microfinance institutions from covering their costs, which chokes off the supply of credit.
10. Microfinance institutions should measure and disclose their performance – both financially and socially.

Microfinance is considered as a tool for socio-economic development, and can be clearly distinguished from charity. Families who are destitute, or so poor they are unlikely to be able to generate the cash flow required to repay a loan, should be recipients of charity. Others are best served by financial institutions.

Debates at the boundaries

There are several key debates at the boundaries of microfinance. Practitioners and donors from the charitable side of microfinance frequently argue for restricting microcredit to loans for productive purposes–such as to start or expand a microenterprise. Those from the private-sector side respond that because money is fungible, such a restriction is impossible to enforce, and that in any case it should not be up to rich people to determine how poor people use their money.

Perhaps influenced by traditional Western views about usury, the role of the traditional moneylender has been subject to much criticism, especially in the early stages of modern microfinance. As more poor people gained access to loans from microcredit institutions however, it became apparent that the services of moneylenders continued to be valued. Borrowers were prepared to pay very high interest rates for services like quick loan disbursement, confidentiality and flexible repayment schedules. They did not always see lower interest rates as adequate compensation for the costs of attending meetings, attending training courses to qualify for disbursements or making monthly collateral contributions.

They also found it distasteful to be forced to pretend they were borrowing to start a business, when they were often borrowing for other reasons (such as paying for school fees, dealing with health costs or securing the family food supply) (Robert ,1989) The more recent focus on inclusive financial systems (see section below) affords moneylenders more legitimacy, arguing in favour of regulation and efforts to increase competition between them to expand the options available to poor people.

Modern microfinance emerged in the 1970s with a strong orientation towards private-sector solutions. This resulted from evidence that state-owned agricultural development banks in developing countries had been a monumental failure, actually undermining the development goals they were intended to serve (Adams et al ,1984). Nevertheless public officials in many countries hold a different view, and continue to intervene in microfinance markets.

There has been a long-standing debate over the sharpness of the trade-off between 'outreach' (the ability of a microfinance institution to reach poorer and more remote people) and its 'sustainability' (its ability to cover its operating costs—and possibly also its costs of serving new clients—from its operating revenues) (Adrian & Richard ,2006). Although it is generally agreed that microfinance practitioners should seek to balance these goals to some extent, there are a wide variety of strategies, ranging from the minimalist profit-orientation of BancoSol in Bolivia to the highly integrated not-for-profit orientation of BRAC in Bangladesh. This is true not only for individual institutions, but also for governments engaged in developing national microfinance systems.

Microfinance experts generally agree that women should be the primary focus of service delivery. Evidence shows that they are less likely to default on their loans than men. Industry data from 2006 for 704 MFIs reaching 52 million borrowers includes MFIs using the solidarity lending methodology (99.3% female clients) and MFIs using individual lending (51% female clients). The delinquency rate for solidarity lending was 0.9% after 30 days (individual lending—3.1%), while 0.3% of loans were written off (individual lending—0.9%) (MIX, 2007). Because operating margins become tighter the smaller the loans delivered, many MFIs consider the risk of lending to men to be too high. This focus on women is questioned sometimes, however. A recent study of microenterpreneurs from Sri Lanka published by the World Bank found that the return on capital for male-owned businesses (half of the sample) averaged 11%, whereas the return for women-owned businesses was 0% or slightly negative (Mckenzie & David, 2008). Microfinancial services may be needed everywhere, including the developed world. However, in developed economies intense competition within the financial sector, combined with a diverse mix of different types of financial institutions with different missions, ensures that most people have access to some financial services. Efforts to transfer microfinance innovations such as solidarity lending from developing countries to developed ones have met with little success (Cheryl, 2001).

Financial Needs and Financial Services.

In developing economies and particularly in the rural areas, many activities that would be classified in the developed world as financial are not monetized: that is, money is not used to carry them out. Almost by definition, poor people have very little money. But circumstances often arise in their lives in which they need money or the things money can buy.

In Stuart Rutherford's recent book *The Poor and Their Money*, he cites several types of needs (Rutherford, 2000):

- *Lifecycle Needs*: such as weddings, funerals, childbirth, education, homebuilding, widowhood, old age.
- *Personal Emergencies*: such as sickness, injury, unemployment, theft, harassment or death.

- *Disasters*: such as fires, floods, cyclones and man-made events like war or bulldozing of dwellings.
- *Investment Opportunities*: expanding a business, buying land or equipment, improving housing, securing a job (which often requires paying a large bribe), etc.

Poor people find creative and often collaborative ways to meet these needs, primarily through creating and exchanging different forms of non-cash value. Common substitutes for cash vary from country to country but typically include livestock, grains, jewelry, and precious metals.

As Marguerite Robinson (2001, p. 54) describes in The Microfinance Revolution, the 1980s demonstrated that "microfinance could provide large-scale outreach profitably," and in the 1990s, "microfinance began to develop as an industry". In the 2000s, the microfinance industry's objective is to satisfy the unmet demand on a much larger scale, and to play a role in reducing poverty. While much progress has been made in developing a viable, commercial microfinance sector in the last few decades, several issues remain that need to be addressed before the industry will be able to satisfy massive worldwide demand.

The obstacles or challenges to building a sound commercial microfinance industry include:

- Inappropriate donor subsidies
- Poor regulation and supervision of deposit-taking MFIs
- Few MFIs that meet the needs for savings, remittances or insurance
- Limited management capacity in MFIs
- Institutional inefficiencies
- Need for more dissemination and adoption of rural, agricultural microfinance methodologies

Current Scale of Microfinance Operations

No systematic effort to map the distribution of microfinance has yet been undertaken. A useful recent benchmark was established by an analysis of 'alternative financial institutions' in the developing world in 2004 (Robert,2004). The authors counted approximately 665 million client accounts at over 3,000 institutions that are serving people who are poorer than those served by the commercial banks. Of these accounts, 120 million were with institutions normally understood to practice microfinance. Reflecting the diverse historical roots of the movement, however, they also included postal savings banks (318 million accounts), state agricultural and development banks (172 million accounts), financial cooperatives and credit unions (35 million accounts) and specialized rural banks (19 million accounts).

Regionally the highest concentration of these accounts was in India (188 million accounts representing 18% of the total national population). The lowest concentrations were in Latin American and the Caribbean (14 million accounts representing 3% of the total population) and Africa (27 million accounts representing 4% of the total population, with the highest rate of penetration in West Africa, and the highest growth rate in Eastern and Southern Africa (www.mfw4a.org/access-to-finance/microfinance.html). Considering that most bank clients in the developed world need several active accounts to keep their affairs in order, these figures indicate that the task the microfinance movement has set for itself is still very far from finished.

By type of service "savings accounts in alternative finance institutions outnumber loans by about four to one. This is a worldwide pattern that does not vary much by region" (Christen et al .2004).

An important source of detailed data on selected microfinance institutions is the *MicroBanking Bulletin*, which is published by Microfinance Information

Exchange. At the end of 2009 it was tracking 1,084 MFIs that were serving 74 million borrowers ($38 billion in outstanding loans) and 67 million savers ($23 billion in deposits) (MIX ,2009).

As yet there are no studies that indicate the scale or distribution of 'informal' microfinance organizations like ROSCA's and informal associations that help people manage costs like weddings, funerals and sickness. Numerous case studies have been published however, indicating that these organizations, which are generally designed and managed by poor people themselves with little outside help, operate in most countries in the developing world.

Help can come in the form of more and better qualified staff, thus higher education is needed for microfinance institutions. This has begun in some universities, as Oliver Schmidt describes.

"Inclusive Financial Systems"

The microcredit era that began in the 1970s has lost its momentum, to be replaced by a 'financial systems' approach. While microcredit achieved a great deal, especially in urban and near-urban areas and with entrepreneurial families, its progress in delivering financial services in less densely populated rural areas has been slow.

The new financial systems approach pragmatically acknowledges the richness of centuries of microfinance history and the immense diversity of institutions serving poor people in developing world today. It is also rooted in an increasing awareness of diversity of the financial service needs of the world's poorest people, and the diverse settings in which they live and work.

Brigit Helms in her book 'Access for All: Building Inclusive Financial Systems', distinguishes between four general categories of microfinance providers, and argues for a pro-active strategy of engagement with all of them to help them achieve the goals of the microfinance movement.

Informal Financial Service Providers

These include moneylenders, pawnbrokers, savings collectors, money-guards, ROSCAs, ASCAs and input supply shops. Because they know each other well and live in the same community, they understand each other's financial circumstances and can offer very flexible, convenient and fast services. These services can also be costly and the choice of financial products limited and very short-term. Informal services that involve savings are also risky; many people lose their money.

Member-Owned Organizations

These include self-help groups, credit unions, and a variety of hybrid organizations like 'financial service associations' and CVECAs. Like their informal cousins, they are generally small and local, which means they have access to good knowledge about each others' financial circumstances and can offer convenience and flexibility. Since they are managed by poor people, their costs of operation are low. However, these providers may have little financial skill and can run into trouble when the economy turns down or their operations become too complex. Unless they are effectively regulated and supervised, they can be 'captured' by one or two influential leaders and the members can lose their money.

NGOs

The Microcredit Summit Campaign counted 3,316 of these MFIs and NGOs lending to about 133 million clients by the end of 2006. Led by Grameen Bank and BRAC in Bangladesh, Prodem in Bolivia, and FINCA International, headquartered in Washington, DC, these NGOs have spread around the developing world in the past three decades; others, like the Gamelan Council, address larger regions. They have proven very innovative, pioneering banking techniques like solidarity lending, village banking and mobile banking that have overcome barriers to serving poor populations. However, with boards that don't necessarily represent either their capital or their customers, their governance structures can be fragile, and they can become overly dependent on external donors.

Formal Financial Institutions

In addition to commercial banks, these include state banks, agricultural development banks, savings banks, rural banks and non-bank financial institutions. They are regulated and supervised, offer a wider range of financial services, and control a branch network that can extend across the country and internationally. However, they have proved reluctant to adopt social missions, and due to their high costs of operation, often can't deliver services to poor or remote populations.

With appropriate regulation and supervision, each of these institutional types can bring leverage to solving the microfinance problem. For example, efforts are being made to link self-help groups to commercial banks, to network member-owned organizations together to achieve economies of scale and scope, and to support efforts by commercial banks to 'down-scale' by integrating mobile banking and e-payment technologies into their extensive branch networks.

Evidence for Reducing Poverty

Some proponents of microfinance have asserted, without offering credible evidence, that microfinance has the power to single-handedly defeat poverty. This assertion has been the source of considerable criticism. Research on the actual effectiveness of microfinance as a tool for economic development remains slim, in part owing to the difficulty in monitoring and measuring this impact (Littlefied et al. 2003).At the 2008 Innovations for Poverty Action/Financial Access Initiative Microfinance Research conference, economist Jonathan Morduch of New York University noted there are only one or two methodologically sound studies of microfinance's impact.

The BBC *Business Weekly* program reported that much of the supposed benefits associated with microfinance, are perhaps not as compelling as once thought. In a radio interview with Professor Dean Karlan of Yale University, a point was raised concerning a comparison between two groups: one African, financed through microcredit and one control group in the Philippines. The results of this study suggest that many of the benefits from microcredit are in fact loaned to people with existing business, and not to those seeking to establish new businesses. Many of those receiving microcredit also used the loans to supplement the family income. The income that went up in business was true only for men, and not for women. This is striking because one of the supposed major beneficiaries of microfinance is supposed to be targeted at women. Professor Karlan's conclusion was that whilst microcredit is not necessarily bad and can generate some positive benefits, despite some lenders charging interest rates between 40-60%, it isn't the panacea that it is purported to be. He advocates rather than focusing strictly on microcredit, also giving citizens in poor countries access to rudimentary and cheap savings accounts.

To further the point stated by Prof Karlan , microfinancing begets the general tendency of a small business initially supported on credit to gain profits with time and generate micro savings. In his latest study, the famous two time Pulitzer Prize winner, Nicholas Donabet Kristof states that there is no evidence of any negative influence of micro financing but countless examples of people now looking at the bigger picture and saving for better things have surfaced. The example of BancoSol (Bolivia), where the number of savers has grown to twice as much as the number of borrowers, further strengthens his theory.

Sociologist Jon Westover found that much of the evidence on the effectiveness of microfinance for alleviating poverty is based in anecdotal reports or case studies.

He initially found over 100 articles on the subject, but included only the 6 which used enough quantitative data to be representative, and none of which employed rigorous methods such as randomized control trials similar to those reported by Innovations for Poverty Action and the M.I.T. Jameel Poverty Action Lab. One of these studies found that microfinance reduced poverty. Two others were unable to conclude that microfinance reduced poverty, although they attributed some positive effects to the program. Other studies concluded similarly, with surveys finding that a majority of participants feel better about finances with some feeling worse (Westover, 2008)

Microfinance and Social Interventions

There are currently a few social interventions that have been combined with micro financing to increase awareness of HIV/AIDS. Such interventions like the "Intervention with Microfinance for AIDS and Gender Equity" (IMAGE) which incorporates microfinancing with "The Sisters-for-Life" program a participatory program that educates on different gender roles, gender-based violence, and HIV/AIDS infections to strengthen the communication skills and leadership of women (Kim et al.2007). "The Sisters-for-Life" program has two phases where phase one consists of ten one-hour training programs with a facilitator with phase two consisting of identifying a leader amongst the group, train them further, and allow them to implement an Action Plan to their respective centres .

Microfinance has also been combined with business education and with other packages of health interventions (Stephen, 2002). A project undertaken in Peru by Innovations for Poverty Action found that those borrowers randomly selected to receive financial training as part of their borrowing group meetings had higher profits, although there was not a reduction in "the proportion who reported having problems in their business" (Karlan ,2009).

Evolution of the Microfinance Sub-Sector in Ghana.

Indeed, the concept of microfinance is not new in Ghana, There has always been in tradition of people saving and/or taking small loan from individuals and groups within the context of self-help to start businesses or farming ventures. For example, available evidence suggests that the first credit union in Africa was established in Northern Ghana in 1955 by Canadian Catholic missionaries. However, susu , which is one of the microfinance schemes in Ghana, is thought to have originated from Nigeria and spread to Ghana in the early twentieth century.

Over the years, the microfinance sector has thrived and evolved into its current state thanks to various financial sector policies and programmes undertaken by different governments since independence. Among these are:

Provisionofsubsidizedcreditsin1950s;

Establishment of the Agriculture Development Bank in 1965 specifically to address the financial needs of fisheries and agriculture sector;

Establishment of Rural and commercial Banks (RCBs), and the introduction of regulations such as commercial banks being required to set aside 20% of total portfolio, to promote lending to agriculture and small scale industries in the 1970s and early1980s;

Shifting from a restrictive financial sector regime to a liberalized regime in 1986;

Promulgation of PNDC Law 328 in 1991 to allow the establishment of different categories of non-bank financial institutions, including savings and loans companies and credit unions.

The policies have led to the emergence of three broad categories of microfinance institutions. These are;

Formal suppliers such as savings and loans companies, rural and community banks, as well as some development and commercial banks; semi-formal supplies such as credit unions, financial non-governmental organizations (FNGOs), and co-operatives;

Informal suppliers such as susu collectors and clubs rotating and accumulating savings and credit associations (ROSCAs and ASCAs), traders, moneylenders and other individuals.

In terms of the regulatory framework, rural and community banks are regulated under the Banking Act 2004 (Act 673), while the savings and loans companies are currently regulated under the Non-Bank Financial institutions (NBFI) Law 1993 (PNDCL 328)

On the other hand the regulatory framework for credit union is now being prepared, and this would recognize their dual nature as co-operative and institutions.The rest of the players such as FNGOs, ROSCAS. And ASCAS do not have legal and regulatory frameworks.

Programmes currently addressing the sub-sector in Ghana include the Financial sector improvement project, Financial sector strategic plan (FINSSP), the Rural Financial services project (RESP), the United Nation Development Programme (UNDP),

Microfinance Project, the social investment Fund (SIF), the community Based Rural

Development Programme (CBRDP), Rural Enterprise Project (REP), and Agricultural services Investment project (ASSIP)

Microfinance and Poverty Reduction in Ghana

The main goal of Ghana and poverty Reduction Strategy (GPRS II) is to ensure "Sustainable equitable growth, accelerated poverty reduction and the vulnerable and excluded within a decentralized, democratic environment"

The intention is to eliminate widespread poverty and growing income inequality, especially among the productive poor who constitute the majority of the working population

According to the 2000 population and Housing census, 80% of the populations are found in the private informal sector.

This group is characterized by lack of access to credit, which constrains the development and growth of that sector of the economy. Clearly, access to financial services is imperative for the development of the informal sector and also helps to mop up excess liquidity through savings that can be made available as investment capital for natural development. Unfortunately, in spite of the obvious roles that microfinance institutions have been playing in the economy particularly over the last twenty years, there is lack of data on their operations

It is known that loan advanced by microfinance institutions are normally for purposes such as housing, petty trade, and “start up” loans for farmers to buy inputs for farming and this includes rice seed, fertilizers and other agricultural tools.

Some of the loans are used for a variety of non-crop activities such as: dairy cow raising , cattle fattening, poultry farming, weaving, basket making leasing farm and other capital machinery and wood working. Of course, funds may be used for a number of other activities, such as crop and animal trading, cloth

trading and pottery manufacture. There are other instances where credit is given to groups consisting of a number of borrowers for collective enterprise, such as : Irrigation pumps, building sanitary latrines, power loans, leasing markets or leasing lad for co-operative farming.

For example, trends in loans and advances extended to small business, individuals and groups by the Non-Bank Financial institutions(NBFIs) in Ghana amounted GH¢50.97 million in 2002 as against GH¢39.64 million in 2001, indicating about 28.6% growth. The amount of loans extended by NBFIs further increased from GH¢70.63 million in 2003 to GH¢72.85 million in 2004, suggesting 3.1% growth. In 2006 alone, total of GH¢160.47 million was extended to clients, which represent48.8% higher then the previous year's total loans and advances granted by these microfinance institutions. The upward-trending NBFIs credit to individuals, small businesses, groups and other indicates marked improvements in level of microfinance in the country.

The Rural and community banks also play very important role in microfinance in the country.

These banks were established specifically to advance loans to small enterprise, farmers, individuals and others within their catchment area. Total loans advanced to clients by all community and rural banks in Ghana was GH¢20.68 million in 2002 compared to GH¢13.12 million in 2001, suggesting an increase of 28.6%.

Profile of Western Microfinance Ltd

Western Microfinance Ltd was established in 2002 to encourage savings and also help the fishermen of Sekondi and Takoradi to access small loans. The head office is at Takoradi . The company within its ten years of operations has provided services to the Sekondi- Takoradi Metropolitan Area, Shama and Mpohor Wassa

East Districts in the Western region. It has opened up agencies and mobilization canters to bring services to five communities in its catchment area. The main occupations of the catchment area are fishing, farming and cottage industry. The Western Microfinance Ltd has a vision to be the leading Microfinance Company with Community Development in it Catchment area and a mission to be microfinance of choice in Ghana through efficient management with innovative customized products to gain greater market share.

1.2 Problem Statement.

In Ghana, due to poor allocation of funds some microfinance institutions record marginal profits with some running at a lost leading to the collapse of the institution. The changing face of the Microfinance Institution coupled with the need to sustain and improve its performance especially when many have collapsed in the past is necessary that continued relevance of existing models be evaluated.

1.3 Objective

The main objective of this study is to examine how Western Microfinance Ltd disburses their funds for loans and formulate it as linear programming problem and solved to:

(1) Determine optimal solution for their allocation of funds for loan disbursement leading to financial sustainability.

(2) Maximize profits on loans

(3) Make recommendations that can address the issue of loan portfolio in the Microfinance Industry

1.4 Methodology

In order for the microfinance institution to maximize their profits, the proposed model will be based on their loan policy and its previous history on loan disbursement.

The model will be solved using the Revised Simplex method. The linear programming model has 3 basic components that is the objective function which is to maximized, the constraints or limitation and the non-negativity constraints.

1.5 Justification

Linear programming models are important tools for financial and Microfinance Institutions. The absence of a trusted model to help disburse funds allocated for loans has led to the frustrations and collapse of microfinance institutions in an economy such as Ghana. If loan limitations are not revised when circumstances change, a microfinance institution could be operating within guidelines that are too restrictive and if guidelines do not comply with current laws and rules, lending decisions may not reflect best practices or regulatory requirement.

A loan policy that does anticipate risks can lead to asset quality problem and poor earnings. The Microfinance Institution might run at a lost or even collapse if they are not able to retrieve all the loans they give out. Due to this, a more scientific and mathematical methods must be used to ensure adequate, effective and efficient distribution of funds they have available for loans to ensure constant growth and sustainability of the Institution.

The proposed model will help Microfinance Institutions to efficiently distribute their funds for loans in order to maximize profit margin. The proposed model will also help decision makers to formulate prudent and effective loan policies

1.6 Organization of the Thesis

The study will be presented in five (5) chapters. Chapter 1 is the Background of the study, Chapter 2 gives will feature the literature review, Chapter 3 gives the methodology. Chapter 4 will dwell on data analysis and discussion. Chapter 5 will highlight on the conclusion and recommendations.

CHAPTER 2

Literature Review

In this section of the work, other people's works, journals of various fields of research on linear programming programs and portfolio theory will be considered.

2.1 Linear Programming Models

Linear programming theory and technique have been successful applied to various transportation problems almost since its early beginning. A famous example is given by Dantzig to adapt his simplex method to solve Hitchcock`s transportation problem. The terminology, such as transportation/assignment problems, and have become a standard in these contexts since then. Linear programming methods were first used to study 0rigin-Destination distributions in 1970s.

2.2 Linear Programming in Financial Management

The use of linear and other types of mathematical programming techniques has received

coverage in the extensive banking literature. Chambers and Chames (1961), as well as Cohen and Hammer (1967;1972), developed a series of sophisticated linear programming models for managing the balance sheet of larger banks, while Waterman and Gee (1963) and Fortson and Dince (1977) proposed less elegant formulations which were better suited for the small to medium-sized bank. Several programming models have also been proposed for managing a bank's investment security portfolio, including those by Booth (1972).

.2.3 Modern Portfolio Theory

According to Bodie et al., (2009), the concept of investment diversification is an old one and existed long before modern finance theory. It was, however, not until 1952 that Harry Markowitz published a formal model of portfolio selection based on diversification principles. This work contributed to Markowitz receiving the Nobel Prize in Economics in 1990. His model can be regarded a first step in portfolio management, which is the identification of the efficient set of portfolios or *the efficient frontier of risky assets.*

Actually, the work began in 1900 when the French mathematician, Louis Bachelier, studied financial markets. Based on his studies, Bachelier argued that prices will go up or down with equal probability and that their volatility is measurable. The so-called bell curve was born, whereby the distribution of price movements is thought to be bell-shaped with very large changes assumed to be extremely rare. It was Markowitz who took the first step in applying Bachelier's ideas (Mandelbrot , 2004).

2.4 Markowitz's portfolio theory

In the 1950's the investment community talked about risk but there was no measurable specification for the term. However, investors were eager to quantify their risk variable. Markowitz showed that the variance of the rate of return was an important measure of risk under a reasonable set of assumptions and came forward with the formulas for computing the variance of the portfolio. The use of this formula revealed the importance of diversifying to reduce risk and also provided guidance on how to diversify effectively (Reilly, 1989).

When Markowitz first published his ideas of portfolio selection in 1952 he rejected the notion that investors should maximize discounted returns and choose their portfolio accordingly. Markowitz's view was that this rule failed to imply diversification, no matter how the anticipated returns were formed. The rule he rejected implied that the investor should place all of his or hers funds in the security with the greatest discounted value. He also rejected the law of large numbers in portfolios made up of securities, objecting to the claim that it would result in both maximum expected returns and minimum variance, and pointing out that returns from securities are too intercorrelated for all variance to be eliminated with diversification. Markowitz also pointed out that a portfolio with maximum expected returns is not necessarily the one with the minimum variance. Hence, that there is a rate at which the investor can gain expected returns by accepting more variance, or reduce variance by giving up expected returns. Building on these observations he presented the "expected returns-variance of returns" rule (Markowitz, 1952). Markowitz‘s idea was that investors should hold mean-variance efficient portfolios. While not an entirely new concept, mean-variance optimization was not a widely used strategy at the time. Most investment managers were focusing their efforts on identifying securities with high expected returns (Chan, Karceski, & Lakonishok, 1999).

In his paper, Markowitz formally presented his view that although investors want to maximize returns on securities they also want to minimize uncertainty, or risk. These are conflicting objectives which must be balanced against each other when the investor makes his or her decision. Markowitz asserts that investors should base their portfolio decisions only on expected returns, i.e. the measure of potential rewards in any portfolio, and standard deviation, the measure of risk. The investor should estimate the expected returns and standard deviation of each portfolio and then choose the best one on the grounds of the relative magnitudes of these two parameters (Sharpe, Alexander, & Bailey, 1999).

As previously mentioned, Markowitz rejected the expected returns rule on the grounds that it neither acknowledged nor accounted for the need for diversification, contrary to his “expected return-variance of return” rule. In addition, he concluded that the expected return-variance of return rule not only revealed the benefits of diversification but that it pointed towards the right type of diversification for the right reason. It is not enough to diversify by simply increasing the number of securities held. If, for example, most of the firms in the portfolio are within the same industry they are more likely to do poorly at the same time than firms in separate industries.

In the same way it is not enough to make variance small to invest in large number of securities. It should be avoided to invest in securities with high covariance among themselves and it is obvious that firms in different industries have lower covariance than firms within the same industry (Markowitz, 1952). Simply put, Markowitz concluded that by mixing stocks that flip tail and those that flip heads you can lower the risk of your overall portfolio. If you spread your investments across unrelated stocks you will maximize your potential profit whether the economy is slowing down or growing. If you then add more and more stock in different combinations you have what Markowitz called an ‘efficient‘portfolio. An efficient portfolio is the portfolio which gives the highest profit with the least risk. The aim of Markowitz‘s methods is to construct that kind of portfolio (Mandelbrot, 2004). Until Markowitz suggested this approach to portfolio analysis no full and specific basis existed to justify diversification in portfolio selection. Also the concept of risk had rarely been defined in a thorough manner

in portfolio analysis before Markowitz's writings, let alone treated analytically. With his approach these issues, diversification and risk, got a specified framework and a workable algorithm for employing that framework for practical problems was provided. Markowitz did not, however, suggest a preferred technique for security analysis or a suitable method for portfolio selection. He concentrated on providing a general structure for the whole process and providing an algorithm for performing the task of portfolio analysis (Sharpe W. F., Portfolio Analysis, 1967).
Markowitz created a theory of portfolio choice in the uncertain future. He quantified the difference between the risk that was taken on individual assets and the aggregated risk of the portfolio. He showed that the portfolio risk came from covariances of the assets which made up the portfolio. The marginal contribution of a security to the portfolio return variance is therefore measured by the covariance between the return of the security and the return of the portfolio but not by the variance of the security itself. In his writings, Markowitz argues that the risk of a portfolio is less than the risk of each asset in the portfolio taken individually and provides quantitative evidence of the merits of diversification (Amenc & Le Sourd, 2003).

In his model of portfolio management Markowitz indentified the efficient set of portfolios, or the efficient "frontier of risky assets". The principal idea behind the frontier set of risky portfolios is that the investor should only be interested in the portfolio which gives the highest expected return for any given risk level. Also, the frontier is a set of portfolios that minimizes the variance for any target expected return (Bodie, Kane, & Marcus, 2009).

With his work, Markowitz introduced a parametric optimization model that was both sufficiently general to be applicable to a significant range of practical situations and simple enough to be usable for theoretical analysis. Nevertheless, the subject is so complicated that Markowitz's work in the 1950's probably raised more questions than it answered. Indeed, it spurred a tremendous amount of related research (Steinbach, 2001).

CHAPTER 3

METHODOLOGY

This chapter takes a critical look at the methodology adopted for the study. We shall discuss linear programming with particular emphases on the revised simplex method.

3.1 Convex Optimization

Convex Sets

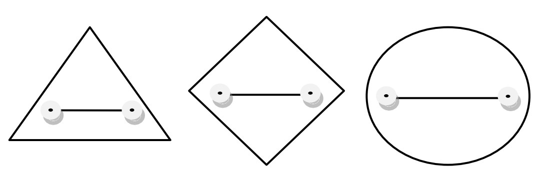
A line segment joining the points x_1 and x_2 in R^n is the set

$$[x_1, x_2] = \{x \in R^n : x = \lambda x_1 + (1 - \lambda) x_2, 0 \leq \lambda \leq 1\}$$

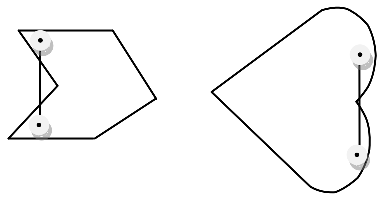
A point on the line segment for which $0 < x < 1$, is called an interior point of the line segment.

A subset S of R^n is said to be convex if for any two elements x_1 x_2 in S, the line segment $[x_1 x_2]$ is contained in S. Thus x_1 and x_2 in S imply $\lambda x_1 + (1-\lambda)x_2 \in S$ for all $0 \leq \lambda \leq 1$ if S is convex.

As examples, we note that in the plane, the following sets are convex.



whereas the Sets



are not Convex

Remarks

i) Generally speaking we may know observe that sets in R^n are convex if they contain no "hole", "indentations" or "Protrusion" and are non-convex otherwise.

ii) The intersection of any family of convex sets in R^n is convex.

iii) A closed half space or open half – space in R^n is convex.Hence a hyperplane, being the intersection of two closed half space is convex.

iv) If A is an m x n matrix and b is an m-vector, then the set of solution of the linear system

Ax = b,

being the intersection of a finite number of hyperplanes in R^n, is convex, Hence the set of all x satisfying the condition Ax = b, $x \geq 0$, is convex, since it is the intersection of a convex set and a half – space, which is convex.

A point u in a non-empty convex set S is said to be an extreme point S if it is not an interior point of any line segment in S. Hence u is an extreme point of u if there are no two distinct point x_1 and x_2, of S such that u =λ x_1 + (1 – λ)x_2 , 0 < λ < 1

Equivalently, u is an extreme point of S whenever u = λ x_1 + (1 – λ) x_2 for x_1 , x_2 in S

and 0 ≤ λ ≤ 1, then $x_1 = x_2 = u$

Convex minimization, a subfield of optimization, studies the problem of minimizing convex functions over convex sets. The convexity property can make optimization in some sense "easier" than the general case - for example, any local optimum must be a global optimum.

Given a real vector space *X* together with a convex, real-valued function

$$f : \mathcal{X} \to \mathbb{R}$$

defined on a convex subset $\mathcal{X}$ of *X*, the problem is to find a point x^* in $\mathcal{X}$ for which the number *f*(*x*) is smallest, i.e., a point x^* such that

$$f(x^*) \leq f(x) \text{ for all } x \in \mathcal{X}.$$

The convexity of *f* makes the powerful tools of convex analysis applicable: the Hahn–Banach theorem and the theory of subgradients lead to a particularly satisfying theory of necessary and sufficient conditions for optimality, a duality theory generalizing that for linear programming, and effective computational methods.

Convex minimization has applications in a wide range of disciplines, such as automatic control systems, estimation and signal processing, communications and networks, electronic circuit design, data analysis and modeling, statistics (optimal design), and finance. With recent improvements in computing and in optimization theory, convex minimization is nearly as straightforward as linear programming. Many optimization problems can be reformulated as convex minimization problems. For example, the problem of *maximizing* a *concave* function *f* can be re-formulated equivalently as a problem of *minimizing* the function -*f*, which is *convex*.

The following statements are true about the convex minimization problem:

- if a local minimum exists, then it is a global minimum.
- the set of all (global) minima is convex.
- for each *strictly* convex function, if the function has a minimum, then the minimum is unique.

These are used by the theory of convex minimization along with geometric notions from functional analysis such as the Hilbert projection theorem, the separating hyperplane theorem, and Farkas' lemma.

The usual and most intuitive form of describing a convex minimization problem consists of the following three parts:

- A **convex function** $f(x) : R^n \longrightarrow R$ to be minimized over the variable x
- **Inequality constraints** of the form $g_i(x) \leq 0$, where the functions g_i are convex
- **Equality constraints** of the form $h_i(x) = 0$, where the functions h_i are affine. In practice, the terms "linear" and "affine" are often used interchangeably. Such constraints can be expressed in the form $h_i(x) = a_i^T x + b_i$, where a_i and b_i are column-vectors..

A convex minimization problem is thus written as

Minimize f(x) subject

Subject to $g_i(x) \leq 0$, $i=1, , , , , m$

$h_i(x) = 0 \quad i=1, , , , ,p$

Note that every equality constraint $h(x) = 0$ can be equivalently replaced by a pair of inequality constraints $h_i(x) \leq 0$ and $-h_i(x) \leq 0$. Therefore, for theoretical purposes, equality constraints are redundant; however, it can be beneficial to treat them specially in practice.

Following from this fact, it is easy to understand why $h_i(x) = 0$ has to be affine as opposed to merely being convex. If $h_i(x)$ is convex, $h_i(x) \leq 0$ is convex, but $-h_i(x) \leq 0$ is *concave*. Therefore, the only way for $h_i(x) = 0$ to be convex is for $h_i(x)$ to be affine.

Methods

Convex minimization problems can be solved by the following contemporary methods:

- "Bundle methods"
- Subgradient projection methods
- Interior-point methods

Other methods of interest:

- Cutting-plane methods
- Ellipsoid method
- Subgradient method

Subgradient methods can be implemented simply and so are widely used.

Maximizing convex functions

Besides convex minimization, the field of *convex optimization* also considers the far more difficult problem of *maximizing* convex functions:

- Consider the restriction of a *convex* function to a compact convex set: Then, on that set, the function attains its constrained maximum only on the boundary. Such results, called "maximum principles", are useful in the theory of harmonic functions, potential theory, and partial differential equations.

Solving even close-to-convex problems can be computationally difficult. The problem of minimizing a quadratic multivariate polynomial on a cube is NP-hard. In fact, in the quadratic minimization problem, if the matrix has only one negative eigenvalue, the problem is NP-hard.

Extensions

Advanced treatments consider convex functions that can attain positive infinity, also; the indicator function of convex analysis is zero for every $x \in \mathcal{X}$ and positive infinity otherwise.

Extensions of convex functions include pseudo-convex and quasi-convex functions. Partial extensions of the theory of convex analysis and iterative methods for approximately solving non-convex minimization problems occur in the field of generalized convexity ("abstract convex analysis").

3.2 Linear Programming

Linear Programming is a method for determining a way to achieve the best outcome (such as maximum profit or lowest cost) in a given mathematical model for some list of requirements represented as linear relationships. Linear programming is a specific case of mathematical programming (mathematical optimization).

More formally, linear programming is a technique for the optimization of a linear objective function, subject to linear equality and linear inequality constraints. Its feasible region is a convex polyhedron, which is a set defined as the intersection of finitely many half spaces, each of which is defined by a linear inequality. Its objective function is a real-valued affine function defined on this polyhedron. A linear programming algorithm finds a point in the polyhedron where this function has the smallest (or largest) value if such point exists.

Linear programs are problems that can be expressed in canonical form:

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} & A\mathbf{x} \leq \mathbf{b} \\ \text{and} & \mathbf{x} \geq \mathbf{0} \end{array}$$

where **x** represents the vector of variables (to be determined), **c** and **b** are vectors of (known) coefficients and *A* is a (known) matrix of coefficients. The expression to be maximized or minimized is called the *objective function* ($\mathbf{c}^{\mathrm{T}}\mathbf{x}$ in this case). The equations $A\mathbf{x} \leq \mathbf{b}$ are the constraints which specify a convex polytope over which the objective function is to be optimized. (In this context, two vectors are comparable when every entry in one is less-than or equal-to the corresponding entry in the other. Otherwise, they are incomparable.)

Linear programming can be applied to various fields of study. It is used most extensively in business and economics, but can also be utilized for some engineering problems. Industries that use linear programming models include transportation, energy, telecommunications, and manufacturing. It has proved useful in modeling diverse types of problems in planning, routing, scheduling, assignment, and design.

Standard Form

Standard form of linear programming consists of the following four parts:

- A **linear function to be maximized**

  e.g $\max_{x1,x2} f(x_1, x_2) = c_1 x_1 + c_2 x_2$

- **Problem constraints** of the following form e.g.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 &\leq b_2 \\ a_{31}x_1 + a_{32}x_2 &\leq b_3 \end{aligned}$$

- **Non-negative variables**

e.g.

$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

- **Non-negative right hand side constants**

$$b_i \geq 0, \; i = 1, 2, 3$$

The problem is usually expressed in *matrix form*, and then becomes:

$$\max\{c^{\mathrm{T}}x \mid 0 \leq Ax \leq b \wedge x \geq 0\}$$

Other forms, such as minimization problems, problems with constraints on alternative forms, as well as problems involving negative variables can always be rewritten into an equivalent problem in standard form.

Augmented Form (Slack Form)

Linear programming problems must be converted into *augmented form* before being solved by the simplex algorithm. This form introduces non-negative *slack variables* to replace inequalities with equalities in the constraints. The problem can then be written in the following block matrix form:

Maximize Z:

$$\begin{bmatrix} 1 & -\mathbf{c}^T & 0 \\ 0 & \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$

$\mathbf{x}, \mathbf{x}_s \geq 0$

where $\mathbf{x}_s$ are the newly introduced slack variables, and Z is the variable to be maximized.

3.3 Duality

Every linear programming problem, referred to as a *primal* problem, can be converted into a dual problem, which provides an upper bound to the optimal value of the primal problem. In matrix form, we can express the *primal* maximum problem as:

Maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq 0$;

with the corresponding **symmetric** dual problem

Minimize $\mathbf{b}^T\mathbf{y}$ subject to $A^T\mathbf{y} \geq \mathbf{c}$, $\mathbf{y} \geq 0$.

There are two ideas fundamental to duality theory. One is the fact that (for the symmetric dual) the dual of a dual linear program is the original primal linear program. Additionally, every feasible solution for a linear program gives a bound on the optimal value of the objective function of its dual.

The weak duality theorem states that the objective function value of the dual at any feasible solution is always greater than or equal to the objective function value of the primal at any feasible solution.

The strong duality theorem states that if the primal has an optimal solution, $\mathbf{x}^*$, then the dual also has an optimal solution, $\mathbf{y}^*$, such that $\mathbf{c}^T\mathbf{x}^*=\mathbf{b}^T\mathbf{y}^*$.

A linear program can also be unbounded or infeasible. Duality theory tells us that if the primal is unbounded then the dual is infeasible by the weak duality theorem. Likewise, if the dual is unbounded, then the primal must be infeasible. However, it is possible for both the dual and the primal to be infeasible.

Complementary slackness

The theorem states:

Suppose that $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ is primal feasible and that $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m)$ is dual feasible. Let $(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m)$ denote the corresponding primal slack variables, and let $(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n)$ denote the corresponding dual slack variables. Then **x** and **y** are optimal for their respective problems if and only if

- $\mathbf{x}_j\mathbf{z}_j = 0$, for $j = 1, 2, \ldots, n$, and
- $\mathbf{w}_i\mathbf{y}_i = 0$, for $i = 1, 2, \ldots, m$.

So if the *i*-th slack variable of the primal is not zero, then the *i*-th variable of the dual is equal zero. Likewise, if the *j*-th slack variable of the dual is not zero, then the *j*-th variable of the primal is equal to zero.

This necessary condition for optimality conveys a fairly simple economic principle:

In standard form (when maximizing), if there is slack in a constrained primal resource (i.e., there are "leftovers"), then additional quantities of that resource must have no value. Likewise, if there is slack in the dual (shadow) price non-negativity constraint requirement, i.e., the price is not zero, then there must be scarce supplies (no "leftovers").

3.4 Existence of Optimal Solutions

Geometrically, the linear constraints define the feasible region, which is a convex polyhedron. A linear function is a convex function, which implies that every local minimum is a global minimum; similarly, a linear function is a concave function, which implies that every local maximum is a global maximum.

Optimal solution need not exist, for two reasons. First, if two constraints are inconsistent, then no feasible solution exists: For instance, the constraints $\mathbf{x} \geq 2$ and $\mathbf{x} \leq 1$ cannot be satisfied jointly; in this case, we say that the LP is *infeasible*. Second, when the polytope is unbounded in the direction of the gradient of the objective function (where the gradient of the objective function is the vector of the coefficients of the objective function), then no optimal value is attained.

Optimal Vertices (And Rays) Of Polyhedra

If a feasible solution exists and if the (linear) objective function is bounded, then the optimum value is always attained on the boundary of optimal level-set, by the *maximum principle* for *convex functions* (alternatively, by the *minimum* principle for *concave functions*): Recall that linear functions are both convex and concave. However, some problems have distinct optimal solutions: For example, the problem of finding a feasible solution to a system of linear inequalities is a linear programming problem in which the objective function is the zero function (that is, the constant function taking the value zero everywhere): For this feasibility problem with the zero-function for its objective-function, if there are two distinct solutions, then every convex combination of the solutions is a solution.

The vertices of the polytope are also called *basic feasible solutions*. The reason for this choice of name is as follows. Let d denote the number of variables. Then the fundamental theorem of linear inequalities implies (for feasible problems) that for every vertex $\mathbf{x}^*$ of the LP feasible region, there exists a set of d (or fewer) inequality constraints from the LP such that, when we treat those d constraints as equalities, the unique solution is $\mathbf{x}^*$. Thereby we can study these vertices by means of looking at certain subsets of the set of all constraints (a discrete set), rather than the continuum of LP solutions. This principle underlies the simplex algorithm for solving linear programs.

3.5 Simplex Algorithm

In mathematical optimization, Dantzig's **simplex algorithm** (or **simplex method**) is a popular algorithm for linear programming. The journal *Computing in Science and Engineering* listed it as one of the top 10 algorithms of the twentieth century.

The name of the algorithm is derived from the concept of a simplex and was suggested by T. S. Motzkin (Murty 1983).Simplices are not actually used in the method, but one interpretation of it is that it operates on simplicial *cones*, and these become proper simplices with an additional constraint. The simplicial cones in question are the corners (i.e., the neighborhoods of the vertices) of a geometric object called polytope. The shape of this polytope is defined by the constraints applied to the objective function.

Figure 3.1 depicts the simplex algorithm beginning at a starting vertex and moving along the edges of the polytope until a optimum solution is reached.

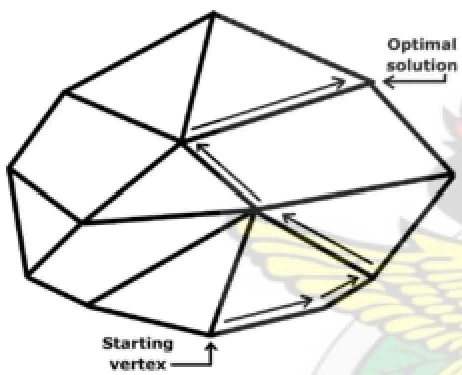


Fig.3.1

A system of linear inequalities defines a polytope as a feasible region. The simplex algorithm begins at a starting vertex and moves along the edges of the polytope until it reaches the vertex of the optimum solution.

The simplex algorithm operates on linear programs in *standard form*, that is linear programming problems of the form,

$$\max_{x1,x2} f(x_1, x_2) = c_1x_1 + c_2x_2$$

Subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &\leq b_1 & & \\ a_{21}x_1 + a_{22}x_2 &\leq b_2 & x_1 \geq 0 & \\ a_{31}x_1 + a_{32}x_2 &\leq b_3 & x_2 \geq 0 & \quad b_i \geq 0,\ i = 1,2,3 \end{aligned}$$

with $x = (x_1, ..., x_n)$the variables of the problem, $c = (c_1, ..., c_n)$are the coefficients of the objective function, *A*, a *p*×*n* matrix and $b = (b_1, ..., b_p)$constants with $b_j \geq 0$. There is a straightforward process to convert any linear program into one in standard form so this results in no loss of generality.

In geometric terms, the feasible region

$$\mathbf{Ax} = \mathbf{b},\ x_i \geq 0$$

is a (possibly unbounded) convex polytope. There is a simple characterization of the extreme points or vertices of this polytope, namely $x = (x_1, ..., x_n)$is an extreme point if and only if the column vectors A_i, where $x_i \neq 0$, are linearly independent. In this context such a point is known as a *basic feasible solution* (BFS).

It can be shown that for a linear program in standard form, if the objective function has a minimum value on the feasible region then it has this value on (at least) one of the extreme points. This in itself reduces the problem to a finite computation since there are finite numbers of extreme points; however the number of extreme points is unmanageably large for all but the smallest linear programs.

It can also be shown that if an extreme point is not a minimum point of the objective function then there is an edge containing the point so that the objective function is strictly decreasing on the edge moving away from the point. If the edge is finite then the edge connects to another extreme point where the objective function has a smaller value, otherwise the objective function is unbounded below on the edge and the linear program has no solution. The simplex algorithm applies this insight by walking along edges of the polytope to extreme points with lower and lower objective values. This continues until the minimum value is reached or an unbounded edge is visited, concluding that the problem has no solution. The algorithm always terminates because the number of vertices in the polytope is finite; moreover since we jump between vertices always in the same direction (that of the objective function), we hope that the number of vertices visited will be small.

The solution of a linear program is accomplished in two steps. In the first step, known as Phase I, a starting extreme point is found. Depending on the nature of the program this may be trivial, but in general it can be solved by applying the simplex algorithm to a modified version of the original program. The possible results of Phase I are either a basic feasible solution is found or that the feasible region is empty. In the latter case the linear program is called *infeasible*. In the second step, Phase II, the simplex algorithm is applied using the basic feasible solution found in Phase I as a starting point. The possible results from Phase II are either an optimum basic feasible solution or an infinite edge on which the objective function is unbounded below.

Slack, Surplus and Artificial Variable

To convert " ≤ " constraints to standard form, a Slack Variable is added to the left hand side of the constraints.

To convert " ≥" constraints to standard form, a Surplus Variable is added to the left hand side of the constraints.

To convert " = , and ≥" constraints, Artificial Variable is used in order to find an initial feasible solution.

The transformation of a linear program to one in standard form may be accomplished as follows. First, for each variable with a lower bound other than 0, a new variable is introduced representing the difference between the variable and bound. The original variable can then be eliminated by substitution. For example, given the constraint

$$x_1 \geq 5$$

a new variable, y_1, is introduced with

$$\begin{aligned} y_1 &= x_1 - 5 \\ x_1 &= y_1 + 5 \end{aligned}$$

The second equation may be used to eliminate x_1 from the linear program. In this way, all lower bound constraints may be changed to nonnegativity restrictions.

Second, for each remaining inequality constraint, a new variable, called a *slack variable*, is introduced to change the constraint to an equality constraint. This variable represents the difference between the two sides of the inequality and is assumed to be nonnegative. For example the inequalities

$$\begin{aligned} x_2 + 2x_3 &\leq 3 \\ -x_4 + 3x_5 &\geq 2 \end{aligned}$$

It is much easier to perform algebraic manipulation on inequalities in this form. In inequalities where ≥ appears such as the second one, some authors refer to the variable introduced as a *surplus variable*.

Are replaced with

$$\begin{aligned} x_2 + 2x_3 + s_1 &= 3 \\ -x_4 + 3x_5 - s_2 &= 2 \\ s_1,\ s_2 &\geq 0 \end{aligned}$$

Third, each unrestricted variable is eliminated from the linear program. This can be done in two ways, one is by solving for the variable in one of the equations in which it appears and then eliminating the variable by substitution. The other is to replace the variable with the difference of two restricted variables. For example if x_2 is unrestricted then we reformulate as

$$x_2 = x_2^{1} - x_2^{11}$$

$$x_2^{1}, x_2^{11} \geq 0$$

The equation may be used to eliminate x_1 from the linear program.

When this process is complete the feasible region will be in the form

$$\mathbf{Ax} = \mathbf{b},\ x_i \geq 0$$

It is also useful to assume that the rank of **A** is the number of rows. This results in no loss of generality since otherwise either the system **Ax** >= **b** has redundant equations which can be dropped, or the system is inconsistent and the linear program has no solution

Canonical Form

A linear programming problem is said to be in canonical form if it has the following structure:

Maximize

$_{j=1}{}^{n}\Sigma\, c_j\ x_j$

Subject to

$_{j=1}{}^{n}\Sigma\, a_{ij}\, x_j \leq b_i$

for each i = 1, , n

$x_j \geq 0$ for each j = 1 , , n

A linear program in standard form can be represented as a *tableau* of the form

$$\begin{bmatrix} 1 & -\mathbf{c}^T & 0 \\ 0 & \mathbf{A} & \mathbf{b} \end{bmatrix}$$

The first row defines the objective function and the remaining rows specify the constraints. If the columns of A can be rearranged so that it contains the identity matrix of order *p* (the number of rows in A) then the tableau is said to be in *canonical form*. The variables corresponding to the columns of the identity matrix are called *basic variables* while the remaining variables are called *nonbasic* or *free variables*. If the nonbasic variables are assumed to be 0, then the values of the basic variables are easily obtained as entries in *b* and this solution is a basic feasible solution.

Conversely, given a basic feasible solution, the columns corresponding to the nonzero variables can be expanded to a nonsingular matrix. If the corresponding tableau is multiplied by the inverse of this matrix then the result is a tableau in canonical form.

Let

$$\begin{bmatrix} 1 & -\mathbf{c}_B^T & -\mathbf{c}_D^T & 0 \\ 0 & I & \mathbf{D} & \mathbf{b} \end{bmatrix}$$

be a tableau in canonical form. Additional row-addition transformations can be applied to remove the coefficients $\mathbf{c}^T{}_B$ from the objective function. This process is called *pricing out* and results in a canonical tableau

$$\begin{bmatrix} 1 & 0 & -\bar{\mathbf{c}}_D^T & z_B \\ 0 & I & \mathbf{D} & \mathbf{b} \end{bmatrix}$$

where z_B is the value of the objective function at the corresponding basic feasible solution. The updated coefficients, also known as *relative cost coefficients*, are the rates of change of the objective function with respect to the nonbasic variables.

Example

Minimize

$Z= -2x -3y-4w$

Subject to

$3x +2y+w\leq 10$

$2x+5y+3w\leq 15$

$x, y, w \geq 0$

With the addition of slack variables s_1 and s_2, this is represented by the canonical tableau

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 & 1 & 0 & 10 \\ 0 & 2 & 5 & 3 & 0 & 1 & 15 \end{bmatrix}$$

where columns 5 and 6 represent the basic variables s_1 and s_2 and the corresponding basic feasible solution is

$x=y=w= 0$, $s_1= 10$, $s_2= 15$

Algorithm

Let a linear program be given by a canonical tableau. The simplex algorithm proceeds by performing successive pivot operations which each give an improved basic feasible solution; the choice of pivot element at each step is largely determined by the requirement that this pivot does improve the solution.

Entering variable selection

Entering variable is the one with the largest objective function value. Since the entering variable will, in general, increase from 0 to a positive number, the value of the objective function will decrease if the derivative of the objective function with respect to this variable is negative. Equivalently, the value of the objective function is decreased if the pivot column is selected so that the corresponding entry in the objective row of the tableau is positive.

If there is more than one column so that the entry in the objective row is positive then the choice of which one to add to the set of basic variables is somewhat arbitrary and several *entering variable choice rules* have been developed which include largest coefficient rule, smallest subscript rule, largest improvement rule and steepest edge rule.

If all the entries in the objective row are less than or equal to 0 then no choice of entering variable can be made and the solution is in fact optimal. It is easily seen to be optimal since the objective row now corresponds to an equation of the form

$$z(\mathbf{x}) = z_B + \text{nonnegative terms corresponding to nonbasic variables}$$

Note that by changing the entering variable choice rule so that it selects a column where the entry in the objective row is negative, the algorithm is changed so that it finds the maximum of the objective function rather than the minimum.

Leaving Variable Selection

Once the pivot column has been selected, the choice of pivot row is largely determined by the requirement that resulting solution will be feasible. First, only positive entries in the pivot column are considered since this guarantees that the value of the entering variable will be nonnegative. If there are no positive entries in the pivot column then the entering variable can take any nonnegative value with the solution remaining feasible. In this case the objective function is unbounded below and there is no minimum.

Next, the pivot row must be selected so that all the other basic variables remain positive. A calculation shows that this occurs when the resulting value of the entering variable is at a minimum. In other words, if the pivot column is *c*, then the pivot row *r* is chosen so that

$$b_r / a_{cr}$$

is the minimum over all r so that $a_{cr} > 0$. This is called the *minimum ratio test*. If there is more than one row for which the minimum is achieved then a *dropping variable choice rule*(the choice between two or more variables tying or entry can be made arbitrarily) can be used to make the determination.

Example

Consider the linear program

Minimize

$Z = -2x - 3y - 4w$

Subject to

$3x + 2y + w \leq 10$

$2x + 5y + 3w \leq 15$

$x, y, w \geq 0$

With the addition of slack variables s and t, this is represented by the canonical tableau

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 & 1 & 0 & 10 \\ 0 & 2 & 5 & 3 & 0 & 1 & 15 \end{bmatrix}$$

Where columns 5 and 6 represent the basic variables *s* and *t* and the corresponding basic feasible solution is

x=y=w= 0 , s= 10 , t= 15

Columns 2, 3, and 4 can be selected as pivot columns, for this example column 4 is selected. The values of *x* resulting from the choice of rows 2 and 3 as pivot rows are 10/1 = 10 and 15/3 = 5 respectively. Of these the minimum is 5, so row 3 must be the pivot row. Performing the pivot produces

$$\begin{bmatrix} 1 & -\frac{2}{3} & -\frac{11}{3} & 0 & 0 & -\frac{4}{3} & -20 \\ 0 & \frac{7}{3} & \frac{1}{3} & 0 & 1 & -\frac{1}{3} & 5 \\ 0 & \frac{2}{3} & \frac{5}{3} & 1 & 0 & \frac{1}{3} & 5 \end{bmatrix}$$

Now columns 4 and 5 represent the basic variables *z* and *s* and the corresponding basic feasible solution is

$$x = y = t = 0,\ z = 5,\ s = 5.$$

For the next step, there are no positive entries in the objective row and in fact

$$Z = -20 + \tfrac{2}{3}x + \tfrac{11}{3}y + \tfrac{4}{3}t$$

so the minimum value of Z is -20.

Finding an Initial Canonical Tableau

In general, a linear program will not be given in canonical form and an equivalent canonical tableau must be found before the simplex algorithm can start. This can be accomplished by the introduction of *artificial variables*. Columns of the identity matrix are added as column vectors for these variables. The new tableau is in canonical form but it is not equivalent to the original problem. So a new objective function, equal to the sum of the artificial variables, is introduced and the simplex algorithm is applied to find the minimum; the modified linear program is called the *Phase I* problem.

The simplex algorithm applied to the Phase I problem must terminate with a minimum value for the new objective function since, being the sum of nonnegative variables, its value is bounded below by 0. If the minimum is 0 then the artificial variables can be eliminated from the resulting canonical tableau producing a canonical tableau equivalent to the original problem.

The simplex algorithm can then be applied to find the solution; this step is called *Phase II*. If the minimum is positive then there is no feasible solution for the Phase I problem where the artificial variables are all zero. This implies that the feasible region for the original problem is empty, and so the original problem has no solution.

Example

Consider the linear program

Minimize

$$Z = -2x - 3y - 4z$$

Subject to

$$\begin{aligned} 3x + 2y + z &= 10 \\ 2x + 5y + 3z &= 15 \\ x,\ y,\ z &\geq 0 \end{aligned}$$

This is represented by the (non-canonical) tableau

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 3 & 2 & 1 & 10 \\ 0 & 2 & 5 & 3 & 15 \end{bmatrix}$$

Introduce artificial variables *u* and *v* and objective function *W* = *u* + *v*, giving a new tableau

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 1 & 0 & 10 \\ 0 & 0 & 2 & 5 & 3 & 0 & 1 & 15 \end{bmatrix}$$

Note that the equation defining the original objective function is retained in anticipation of Phase II. After addition of row 3 and row 4 to remove the coefficients from the objective function this becomes

$$\begin{bmatrix} 1 & 0 & 5 & 7 & 4 & 0 & 0 & 25 \\ 0 & 1 & 2 & 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 1 & 0 & 10 \\ 0 & 0 & 2 & 5 & 3 & 0 & 1 & 15 \end{bmatrix}$$

Select column 5 as a pivot column, so the pivot row must be row 4, and the updated tableau is

$$\begin{bmatrix} 1 & 0 & \frac{7}{3} & \frac{1}{3} & 0 & 0 & -\frac{4}{3} & 5 \\ 0 & 1 & -\frac{2}{3} & -\frac{11}{3} & 0 & 0 & -\frac{4}{3} & -20 \\ 0 & 0 & \frac{7}{3} & \frac{1}{3} & 0 & 1 & -\frac{1}{3} & 5 \\ 0 & 0 & \frac{2}{3} & \frac{5}{3} & 1 & 0 & \frac{1}{3} & 5 \end{bmatrix}$$

Now select column 3 as a pivot column, for which row 3 must be the pivot row, to get

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -\frac{25}{7} & 0 & \frac{2}{7} & -\frac{10}{7} & -\frac{130}{7} \\ 0 & 0 & 1 & \frac{1}{7} & 0 & \frac{3}{7} & -\frac{1}{7} & \frac{15}{7} \\ 0 & 0 & 0 & \frac{11}{7} & 1 & -\frac{2}{7} & \frac{3}{7} & \frac{25}{7} \end{bmatrix}$$

The artificial variables are now 0 and they may be dropped giving a canonical tableau equivalent to the original problem:

$$\begin{bmatrix} 1 & 0 & -\frac{25}{7} & 0 & -\frac{130}{7} \\ 0 & 1 & \frac{1}{7} & 0 & \frac{15}{7} \\ 0 & 0 & \frac{11}{7} & 1 & \frac{25}{7} \end{bmatrix}$$

This is, fortuitously, already optimal and the optimum value for the original linear program is −130/7.

Degeneracy: Stalling and Cycling

If the values of all basic variables are strictly positive, then a pivot must result in an improvement in the objective value. Basic feasible solutions where at least one of the *basic* variables is zero are called *degenerate* and may result in pivots for which there is no improvement in the objective value. In this case there is no actual change in the solution but only a change in the set of basic variables. When several such pivots occur in succession, there is no improvement; in large industrial applications, degeneracy is common and such "*stalling*" is notable. Worse than stalling is that the possibility the same set of basic variables occurs twice, in which case, the deterministic pivoting rules of the simplex algorithm will produce an infinite loop, or "cycle". While degeneracy is the rule in practice and stalling is common, cycling is rare in practice. Bland's rule prevents cycling and thus guarantee that the simplex algorithm always terminates. Bland's rule is an algorithmic refinement of the simplex method which solves feasible linear optimization problems without cycling. Another pivoting algorithm, the criss-cross algorithm never cycles on linear programs.

3.6 The Revised Simplex Method

The original Simplex method is a straight forward algebraic procedure. However, this way of executing the algorithm (in either algebraic or tabular form) is not the most efficient computational procedure for computers because it computes and stores many numbers that are not needed at the current iteration and that may not even become relevant for decision making at subsequent iterations.

The only pieces of information relevant at each iteration are:

1. The coefficients of the non basic variables

2. The coefficients of the entering basic variable in the other equations

3. The right –hand sides of the equations

It would be very useful to have a procedure that could obtain this information efficiently without computing and storing the other coefficients. These considerations motivated the development of the revised simplex method. This method was designed to accomplish exactly the same things as the original simplex method, but in a way that is more efficient for execution on computer. Thus, it is a streamlined version of the original procedure. It computes and stores only the information that is currently needed, and it carries along the essential data in a more compact form.

The revised simplex method explicitly uses matrix manipulations, so it necessary to describe the problem in matrix notation. Using matrices, our standard form for the general linear programming model becomes

Maximize $Z = c\,x$,

Subject to

$Ax \leq b$ and $x \geq 0$,

Where c is the row vector

$C = [C_1, C_2 \ldots\ldots C_n,]$

x, b, are the column vectors such that

$$x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_m \end{bmatrix}$$

and A is the matrix

$$A = \begin{bmatrix} a_1 & a_{12} \, ... \, a,_n \\ a_{21} & a_{22} \quad\quad a_{2n} \\ a_{m1} & a_{m2} \, ... \, a_{mn} \end{bmatrix}$$

To obtain the augmented form of the problem, introduce the column vector of slack variables.

$$x_s = \begin{bmatrix} x_{n+1} \\ x_{n+2} \\ . \\ . \\ x_{n+m} \end{bmatrix}$$

So that the constraints become

$[A, \mathrm{I}] \quad \begin{bmatrix} x \\ xs \end{bmatrix} = b$ and $\begin{bmatrix} x \\ xs \end{bmatrix} \geq 0$ where I is the m x m identify matrix

Solving For a Basic Feasible Solution

Recall that the approach of the simplex method is to obtain a sequence of improving Basic Feasible solutions until on optimal solution is reached. One of the key features of the revised simplex method involves the way in which it solves for each new Basic Feasible solution after identifying its basic and non basic. Given the variables, the resulting basic solution is the solution of the m equations.

$[A, \mathrm{I}] \begin{bmatrix} X \\ Xs \end{bmatrix} = b,$

in which the non basic variables from the n + m elements of

$\begin{bmatrix} x \\ xs \end{bmatrix}$ are set equal to zero. Eliminating these n variables by equating them to zero leaves a set of m equations in m unknowns (the basic variables). This set of equations can be denoted by

$B\,x_B = b,$

Where the vector of basic variables

$$x_B = \begin{bmatrix} x_{B1} \\ x_{B2} \\ \cdot \\ \cdot \\ x_{Bm} \end{bmatrix}$$

is obtained by eliminating the non basic variables from $\begin{bmatrix} x \\ xs \end{bmatrix}$,

and the basis matrix

$$B = \begin{bmatrix} B_{11} & B_{12} \ldots\ldots & B_{1m} \\ B_{21} & B_{22} & B_{2m} \\ \ldots & \ldots & \ldots \\ B_{m1} & B_{m2} & B_{mm} \end{bmatrix}$$

is obtained by eliminating the columns corresponding to coefficients of non basic variables from $\lfloor A, I \rfloor$. (in addition, the elements of x_B and, therefore, the columns of B may be placed in a different order when the simplex method is executed). The simplex method introduces only basic variables such that B is nonsingular, so that B always will exist. Therefore, to solve $B x_B = b$, both sides are premultiplied by B^{-1}.

$B^{-1} B x_B = B^{-1} b$

Since $B^{-1}B = I$, the desired solution for the basic variables is $x_B = B^{-1} b$.

Let C_B be vector whose elements are the objective function coefficients (including Zeros for slack variables) for the corresponding elements of x_B. The value of the objective function for this basic solution is then $Z = C_B x_B = C_B B^{-1}b$

Revised Simplex Algorithm

Original simplex method calculates and stores all numbers in the tableau – many are not needed.

⟶ Revised Simplex Method (more efficient for computing)

Used in all commercial available package. (e. g. IBM MPSX, CDC APEX III)

$$\text{Max } Z = \underline{c}\,\underline{x}$$

$$\text{Subject to } \quad A\,x \leq \underline{b}$$

$$x \geq \underline{0}$$

Initially constraints become (standard form):

$$\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x \\ x_s \end{bmatrix} = \begin{bmatrix} b \end{bmatrix} \qquad \underline{x_s} = \text{slack variables}$$

Basis matrix: columns relating to basic variables.

$$B = \begin{pmatrix} B_{11} & \cdots & B_{1m} \\ \vdots & & \vdots \\ B_{m1} & \cdots & B_{mm} \end{pmatrix}$$

(Initially B = I)

Basic variable values: $x_B = \begin{pmatrix} x_{B1} \\ \vdots \\ x_{Bm} \end{pmatrix}$

At any iteration non – basic variables = 0

$B\, \underline{x}_B = b$

Therefore $x_B = B^{-1} b$ $B^{-1} \longrightarrow$ inverse matrix.

At any iteration, given the original b vector and the inverse matrix, $\underline{x}_B$ (current R.H.S) can be calculated.

$Z = C_B\, x_B$ where C_B = objective coefficients of basic variables.

Steps in the Revised Simplex Method

1. Determine entering variable, X_J with associated vector P_J

-Determine the coefficient of the basic variable, c_B

-Compute $Y = c_B B^{-1}$

-Compute $z_j - c_j = YP_j - c_j$ for all non – basic variables.

Choose largest negative value (maximization)

If none, stop.

2. Determine leaving variable, Xr, with associated vector P_r.

- Compute $x_B = B^{-1}b$ (current R.H.S.)

- Compute current constraint coefficients of entering variable:

$\alpha^j = B^{-1} P_j$

X_r is associated with

$\theta = \text{Min} \left\{ (\underline{x}_B)_k / \alpha^j k,\ \alpha^j{}_k > 0 \right\}$

(Minimum ratio rule)

3. Determine next basis i.e. calculate B^{-1}

Go to step 1.

Example:

Max $Z = 3x_1 + 5x_2$

s. t. $X_1 \leq 4$

$2X_2 \leq 12$

$3X_1 + 2X_2 \leq 18$

$X_1, X_2 \geq 0$

Standard form of constraints: -

$X_1 + S_1 = 4$

$2X_2 + S_2 = 12$

$3X_1 + 2X_2 + S_3 = 18$

$X_1, X_2, S_1, S_2, S_3 \geq 0$

$$x_B = B^{-1} b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix}$$

$C_B = [0 \ 0 \ 0]$

$$Z = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix} = 0$$

First iteration

Step 1

Determine entering variable, X_j, with associated vector P_j.

- Compute $Y = C_B B^{-1}$

- Compute $Z_j - C_j = Y P_j - C_j$ for all non – basic variables.

$$Y = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$P_j = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$Z_1 - C_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - 3 = -3$$

And similarly for $z_2 - c_2 = -5$

Therefore X_2 is entering variable.

Step 2

Determine leaving variable, X_r, with associated vector $P_{r.}$

- Compute $x_B = B^{-1} b$ (current R. H. S.)

- Compute current constraint coefficients of entering variable:

$$\alpha^j = B^{-1} P_j$$

X_r is associated with

$$\theta = \text{Min}\left\{ (\underline{x}_B) / \alpha^j k \;,\; \alpha^j{}_k > 0 \right\}$$

$$\underline{x}_B = \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix} \qquad \alpha^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

$$\theta = \text{Min} \left\{ - , 12/2, 18/2 \right\}$$

$$= 12/2$$

Therefore S_2 leaves the basis.

Step 3

Determine new B^{-1}

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix} \quad B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & ½ & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

Solution after one iteration:

$x_B = B^{-1} b$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & ½ & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 6 \end{pmatrix}$$

Go to step 1

Step 1 (second iteration)

Compute $Y = c_B B^{-1}$

$$Y = \begin{bmatrix} 0 & 5 & 0 \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & ½ & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{bmatrix} 0 & 5/2 & 0 \end{bmatrix}$$

- compute $z_j - c_j = \underline{Y}\,\underline{P}_j - c_j$ for all non – basic variables(X_1 and S_2):

$$X_1: z_1 - c_1 = \begin{bmatrix} 0 & 5/2 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ \\ \end{pmatrix} - 3 = -3$$

$$\begin{matrix} 0 \\ 3 \end{matrix}$$

$$S_2: \; z_4 - c_4 = \begin{bmatrix} 0 & 5/2 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 0 = 5/2$$

Therefore X_1 enters the basis.

Step 2

Determine leaving variable.

$$x_B = \begin{pmatrix} 4 \\ 6 \\ 6 \end{pmatrix} \qquad \alpha^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & ½ & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$\theta = \text{Min}\{4/1, \; - , \; 6/3\}$$

$$= 6/3$$

Therefore S_3 leaves the basis.

Step 3

Determine new B^{-1}

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 1 & 1/3 & -1/3 \\ 0 & ½ & 0 \\ 0 & -1/3 & 1/3 \end{pmatrix}$$

Solution after two iterations:

$x_B = B^{-1} b$

$$= \begin{pmatrix} 1 & 1/3 & -1/3 \\ 0 & 1/2 & 0 \\ 0 & -1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}$$

Go to step 1

Step 1

- Compute Y $= c_B B^{-1}$

$$Y = \begin{bmatrix} 0 & 5 & 3 \end{bmatrix} \begin{pmatrix} 1 & 1/3 & -1/3 \\ 0 & ½ & 0 \\ 0 & -1/3 & 1/3 \end{pmatrix} = [0 \ \ 3/2 \ \ 1]$$

- Compute $z_j - c_j = Y P_j - c_j$ for all non basic

Variables (S_2 and S_3) : -

S_2: $z_4 - c_4 = \begin{bmatrix} 0 & 3/2 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 0 = 3/2$

S_3: $z_5 - c_5 = \begin{bmatrix} 0 & 3/2 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - 0 = 1$

No negatives. Therefore stop.

Optimal solution : $S_1 = 2$ $X_2 = 6$ $X_1 = 2$

$Z^* = \underline{c}_B \underline{x}_B = \begin{bmatrix} 0 & 5 & 3 \end{bmatrix} \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}$

CHAPTER 4

DATA ANALYSIS

In this chapter, we analyze data from Western Microfinance Limited (WMFL). A model is proposed and solved to help maximize its net profit.

The Microfinance Institution is in the process of formulating a loan policy involving a total of GH¢150000. Being a full service facility, the Microfinance Institution (MFI) is obligated to grant loan to different clients.

The table below provides the types of loans, the interest rate charge and the

probability of bad debts of Western Microfinance Ltd(WMFL) as estimated from 2011 financial year.

Types of loan	*Interest rate (r_i)*	*Probability of bad debts (P_i)*
x_1 : Business	0.32	0.15
x_2 : Susu	0.32	0.05

x_3 : Salary	0.30	0.03
x_4 : Farm	0.30	0.21
x_5 : Funeral	0.40	0.02

Bad debts are assumed unrecoverable and hence produce no interest revenue. For policy reasons, there are limits on how the MFI allocates the fund.

The Western Microfinance Ltd requires that the disbursement of fund for loan should be done as follows:

1. Allocates at most 55% of the total funds to Business and Salary loan .

2. Salary loan should be at least 50% of the Susu, Farm and Funeral loans (to ensure optimality).

3. The sum of farm and funeral loans should be at most 15% of the total funds.

4. The sum of Susu and Salary loans should be at least 50% of Business, Farm and Funeral loans.

5. The sum of Business and Farm loans should be at must 30% of the total funds.

6. Farm loans should not exceed 5% of the total funds

7. The total ratio of bad debt on all loans may not exceed 0.5

4.1 Formulation

Let x_1 = Amount for Business loan

x_2 = Amount for Susu loan

x_3 = Amount for Salary loan

x_4 = Amount for Farm loan

x_5 = Amount for Funeral loan

Objective Function

The objective of the MFI is to maximize its net return, Z which comprise of the difference between revenue from interest and lost fund due to bad debts for each amount of loan disburses.

Loan Amount	*Amount of bad debts ($P_i x_i$)*	*Amount Contributing to profit ($(1-P_i) x_i$)*
x_1	0.15 x_1	0.85 x_1
x_2	0.05 x_2	0.95 x_2
x_3	0.03 x_3	0.97 x_3
x_4	0.21 x_4	0.79 x_4
x_5	0.02 x_5	0.98 x_5

Profit on loan is given by

$Z= r_1 (1-P_1) x_1 + r_2 (1-P_2) x_2 + r_3 (1-P_3) x_3 +......$ where $P_i > 0$

The Objective function is

Maximize

$Z = 0.32\ (0.85x_1) + 0.32\ (0.95x_2) + 0.30\ (0.97x_3) + 0.30\ (0.79x_4) + 0.40\ (0.98x_5)$

$Z = 0.272x_1 + 0.304x_2 + 0.291x_3 + 0.237x_4 + 0.392x_5$

Constraints

The problem has Nine (9) constraints

1. Limit on total funds available ($x_{1,}\ x_{2,}\ x_{3,}\ x_{4,}\ x_5$)

The total fund available is GH¢ 150000

$x_1 + x_{2+}\ x_3 + x_4 + x_5 \leq 150000$

2. Limit on Business loans (x_1) and salary loan (x_2)

Allocates at most 55% of the total funds to Business and salary loan

$x_{1+}\ x_3 \leq 0.55\ (150000\ )$

$x_{1\ +\ X3} \leq\ 82500$

3 .Limit on salary (x_3), Susu (x_2), Farm (x_4), and Funeral (x_5) loans

Salary loan should be at least 50% of the Susu, Farm and Funeral loans to ensure optimality.

$x_3 \geq 0.5\ (x_2 + x_4 + x_5)$

$0.5x_2 - x_3 + 0.5x_4 + 0.5x_5 \leq 0$

4. Limit on Farm (x_4) and funeral loans (x_5)

The sum of farm and funeral loans should be of most 15% of the total funds.

$x_4 + x_5 \leq 0.15\ (150{,}000)$

$x_4 + x_5 \leq 22500$

5. Limit on Susu (x_2), Salary(x_3) and Business (x_1), Farm (x_4) and Funeral loans(x_5).

The sum of Susu and salary loans should be at leas 50% of Business, Farm and funeral loans.

$x_2 + x_3 \geq 0.5\ (x_1 + x_4 + x_5)$

$0.5x_1 - x_2 - x_3 + 0.5x_4 + 0.5x_5 \leq 0$

6. Limit on Business (x_1) and Farm loan (x_4)

The Sum of Business and Farm loans should be at most 30% of the total funds.

$$x_1 + x_4 \le 0.3\ (150000)$$

$$x_1 + x_4 \le 45000$$

7. Limit on Farm loan (x_4)

Farm loans should not exceed 5% of the total funds

$$x_4 \le 0.05\ (150000)$$

$$x_4 \le 7500$$

8. Limit on bad debts

The total ratio of bad debt on all loans may not exceed 0.05

$$\frac{0.15x_1 + 0.05x_2 + 0.03x_3 + 21x_4 + 0.02x_5 \le 0.05}{x_1 + x_2 + x_3 + x_4 + x_5}$$

9. Non – negativity

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5 \ge 0$$

Resulting Linear Programming Problem

Maximize

$$Z = 0.272x_1 + 0.304x_2 + 0.291x_3 + 0.237x_4 + 0.392x_5$$

Subject to

$x_1 + x_2 + x_3 + x_4 + x_5 \leq 150000$

$x_1 + x_3 \leq 82500$

$0.5x_2 - x_3 + 0.5x_4 + 0.5x_5 \leq 0$

$x_4 + x_5 \leq 22500$

$0.5x_1 - x_2 - x_3 + 0.5x_4 + 0.5x_5 \leq 0$

$x_1 + x_4 \leq 45000$

$x_4 \leq 7500$

$0.1x_1 - 0.02x_3 + 0.16x_4 - 0.03x_5 \leq 0$

$x_1, x_2, x_3, x_4, x_5, \geq 0$

Qualitative Method

Maximum

$Z = 0.272x_1 + 0.304x_2 + 0.291x_3 + 0.237x_4 + 0.392x_5 + 0s_1, + 0s_2 + 0s_3 + 0s_4$

$+ 0s_5 + 0s_5 + 0s_6 + 0s_7 + 0s_8$

$x_1 + x_2 + x_3 + x_4 + x_5 + s_1 = 150000$

$x_1 + x_3 + s_2 = 82500$

$0.5x_1 – x_3 + 0.5x_4 + 0.5x_5 + s_3 = 0$

$x_4 + x_5 + s_4 = 22500$

$0.5x_1 – x_2 – x_3 + 0.5x_4 + 0.5x_5 + s_5 = 0$

$x_1 + x_4 + s_6 = 45000$

$x_4 + s_7 = 7500$

$0.1x_1 – 0.02x_3 + 0.16x_4 – 0.03x_5 + s_8 = 0$

Computational Procedure

Input Parameters

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0.5 & -1 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1 & 1 \\ 0.5 & -1 & -1 & 0.5 & 0.5 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 150000 \\ 82500 \\ 0 \\ 22500 \\ 0 \\ 45000 \end{pmatrix}$$

0	0	0	1	0	7500
0.1	0	-0.02	0.16	– 0.03	0

Basic variable = [s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8]

Non Basic variable = [x_1 x_2 x_3 x_4 x_5]

Coefficient of Basic, C_B = [0 0 0 0 0 0 0 0]

First Iteration

Step 1

Determine entering variable, X j, with associated Vector P j

-Compute Y= $C_B B^{-1}$

-

Compute Zj – c_j for all non –Basic variables choose (largest negative value)

If none , stop.

B= 1 0 0 0 0 0 0 0 B^{-1}= 1 0 0 0 0 0 0 0

0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0

$$\begin{matrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

$Y = C_B B^{-1}$

$$Y = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0] \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Y = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$$

$Z_j - c_j = YP_j - c_j$

$Z_1 - c_1 = [\,0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\,]$ 1 - 0.304 = -0.304

1

0

0

0.5

1

0

0.1

$Z_1 - c_1 = -\ 0.272$

$Z_2 - c_2 = [\,0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\,]$ - 0.304 = -0.304

1

0

0.5

0

-1

0

0

0

$x_2 : Z_2 - c_2 = - 0.304$

$Z_3 - c_3 = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$ -1 - 0.291 = - 0.291

1

-1

0

-1

0

0

-0.02

$x_3 : Z_{3-}\ c_3 = - 0.291$

$$Z_4 - c_4 = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0] \begin{pmatrix} 1 \\ 0 \\ 0.5 \\ 1 \\ 0.5 \\ 1 \\ 1 \\ 0.16 \end{pmatrix} - 0.237 = -0.237$$

$x_4 : Z_4 - c_4 = -0.237$

$$Z_5 - c_5 = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0] \begin{pmatrix} 1 \\ 0 \\ 0.5 \\ 1 \\ 0.5 \\ 0 \\ 0 \\ -0.03 \end{pmatrix} - 0.392 = -0.392$$

$x_5 : Z_5 - c_5 = -0.392$

Since $Z_5 - c_5$ is the minimum, x_5 enter into Basis.

Step 2

Determine leaving variable, X_r, with associated vector P_r

-Compute $X_B = B^{-1} b$ (Current RHS)

-Compute Current Constraint Coefficients Of entering variable

$\infty^j = B^{-1} P_j$

X r is associated with

$\theta = \text{Min}\{ (XB)_k / \infty^j ,\ \infty^j > o\}$

$X_B = B^{-1} b$

$$X_B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 150000 \\ 82500 \\ 0 \\ 22500 \\ 0 \end{pmatrix} = \begin{pmatrix} 150000 \\ 82500 \\ 0 \\ 22500 \\ 0 \end{pmatrix}$$

0	0	0	0	0	1	0	0	45000	45000
0	0	0	0	0	0	1	0	7500	7500
0	0	0	0	0	0	0	1	0	0

$\infty^j = B^{-1} P_j =$

1	0	0	0	0	0	0	0	1	= 1
0	1	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0.5	0.5
0	0	0	1	0	0	0	0	1	1
0	0	0	0	1	0	0	0	0.5	0.5
0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	1	-0.03	-0.03

θ= Min {150000, - , 0, 22500, 0, __, __, 0 }

Therefore s_4 leaves the Basis since is the minimum

Step 3

Determine new B^{-1}

$$B = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -0.03 & 0 & 0 & 0 & 1 \end{pmatrix} \quad B^{-1} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -0.03 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$X_B = B^{-1} b$

$$X_B = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.03 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 150000 \\ 82500 \\ 0 \\ 22500 \\ 0 \\ 45000 \\ 7500 \\ 0 \end{pmatrix} = \begin{pmatrix} 127500 \\ 82500 \\ -11250 \\ 22500 \\ -11250 \\ 45000 \\ 7500 \\ 675 \end{pmatrix}$$

The Objective function value after one iteration :

$Z = C_B X_B$

$$Z = [0\ 0\ 0\ 0.392\ 0\ 0\ 0\ 0] \begin{pmatrix} 127500 \\ 82500 \\ -11250 \\ 22500 \\ -11250 \\ 45000 \\ 7500 \\ 675 \end{pmatrix} = 8820$$

and x_5=22500, s_1=127500, s_2=82500, s_3= -11250, s_5=-11250, s_6=45000, s_7=7500, s_8=675

Results

Table 4.1 depicts the variables (column one), the optimal value of the variables (column two) and the status of the variables(column three). The variables show that funds for loans should be allocated to Susu, Salary and Funeral loans with the amounts indicated below.

4.2 Optimal Solution: after four (4) iterations

Table 4.1 Optimal Value (Z) = 46930.0

Variable	Value	Status
x_1: Business loan	0.0	Non Basic
x_2: Susu loan	77500.00	Basic
x_3: Salary loan	50000.00	Basic
x_4: Farm loan	0.00	Non Basic
x_5: Funeral loan	22500.00	Basic

Table 4.2 depicts the variables (column one) and objective coefficient (column two) and the objective value contribution (column three).

Table 4.2

Variable	Objective coefficient	Objective value contribution
x_1: Business loan	0.272	0.00
x_2: Susu loan	0.304	23560
x_3: Salary loan	0.291	14550.00
x_4: Farm loan	0.237	0.00
x_5: Funeral loan	0.392	8820
Total		46930

Table 4.3 depicts the constraint (column one) the current right hand side (column two) and the slack-/surplus (column three).

Constraint	Current RHS	Slack - / Surplus
1	150000	0.00
2	82500	32500.00-
3	0.00	0.00
4	22500	0.00
5	0.00	116250.00
6	45000	45000.00
7	7500	7500.00
8	0	1675.00-

Sensitivity Analysis

Table 4.4 depicts the variables (column one), current objective coefficient (column two), the minimum objective coefficient (column three), the maximum objective coefficient (column four) and the reduced cost (column five).

Table 4.4

Variable	Current Objective Coefficient	Minimum Objective Coefficient	Maximum Objective Coefficient	Reduced Cost
x_1: Business loan	0.272	-infinity	0.304	0.030
x_2: Susu loan	0.304	0.291	0.392	0.000
x_3: Salary loan	0.291	0.210	0.304	0.000
x_4: Farm loan	0.237	-infinity	0.392	0.160
x_5: Funeral loan	0.392	0.304	infinity	0.00

Table 4.5 depicts the constraint (column one),the current right hand side (column two),the minimum right hand side (column three),the maximum right hand side (column four) and the dual price (column five).

Table 4.5

Constraint	Current RHS	Min RHS	Max RHS	Dual Price
1	150000	33750.00	24750.00	0.30
2	82500	50000.00	infinity	0.00
3	0.00	-48750.00	7500.00	0.01
4	22500	0.00	100.000.00	0.09
5	0.00	-116250.00	infinity	0.00
6	45000	0.00	infinity	0.00
7	7500	0.00	infinity	0.00
8	0.00	-1675.00	infinity	0.00

DISCUSSION

The reduced cost for x_1, x_3 x_4 and x_5 are 0.03, 0, 0, 0.16 and 0 respectively. The dual price for constraint (1) is 0.30, constraint (3) is 0.01and constraint (4) is 0.09.These are non zero because they correspond to the active constraints at the optimum, hence their slack variable are non-basic (0), so the dual can be non zero.

The optimal solution is x_1 = 0, x_2 = 77500, x_3 = 50000, x_4 = 0 and x_5 = 22500 and the objective function value, is Z = 46930. This shows that the Western Microfinance Ltd should allocate GH¢ 77500 to Susu loans, GH¢ 50000 Salary loans and GH¢22500 to funeral loans and should not allocate fund to business and farm loans.

CHAPTER 5

CONCLUSION

Most microfinance institution in the country do not have any scientific method for allocating of funds for loans and because of that most of them are unable to optimize their profits, thereby leading to the collapse of the institution which affects the economic and social lives of the areas which they operate.

A model has been proposed to help Western Microfinance Limited (WMFI) allocate their funds for loans. Our model shows that if Western Microfinance Ltd (WMFI) sticks to the model they can make an annual profit of GH¢ 46930 on loans as compared to GH¢ 31624 profits made on loans in 2010.

Therefore we conclude that the scientific method used to develop the proposed model can increase Western Microfinance Ltd (WMFI) net profit if they should stick to if.

RECOMMENDATION

We recommend that using mathematical method and scientific methods to give out loans can help Microfinance Institution and all financial institutions to increase their profits. Therefore we recommend Western Microfinance Limited to stick to this model in their allocation of funds for loans.

Again, it is recommended that all Microfinance and other financial institutions should use Mathematical methods and scientific methods in most of their businesses.

The limitations encountered include unpreparedness of the institution to give out data, it should be noted that these models cannot be used to take decision outside Western Microfinance Ltd (WMFL).